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ELEMENTS

OF THE

DIFFERENTIAL AND INTEGRAL CALCULUS,

WITH

EXAMPLES AND APPLICATIONS.

BY

JAMES M. TAYLOR, A.M.,

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PREFACE.

THE object of the following treatise is to present simply and concisely the fundamental problems of the Calculus, their solution, and more common applications.

Since variables are its characteristic quantities, the first fundamental problem of the Calculus is, *To find the ratio of the rates of change of related variables.* To enable the learner most clearly to comprehend this problem, the author has employed the conception of rates, which affords finite differentials and the simplest demonstration of many principles. The problem of Differentiation having been clearly presented, a general method of its solution is obtained by the use of limits. This order of development avoids the use of the indeterminate form $\frac{0}{0}$, and secures all the advantages of the differential notation. Many principles are proved, both by the method of rates and that of limits, and thus each is made to throw light upon the other.

In a final chapter, the method of infinitesimals is briefly presented; its underlying principles having been previously established.

The chapter on Differentiation is followed by one on Integration; and in each, as throughout the work, there are numerous practical problems in Geometry and Mechanics, which serve to exhibit the power and use of the science, and to excite and keep alive the interest of the student.

In writing this treatise, the works of the best American, English, and French authors have been consulted; and from these sources the most of the examples and problems have been obtained.

The author is indebted to Professors J. E. Oliver and J. McMahon of Cornell University, and Professor O. Root, Jr., of Hamilton College, for valuable suggestions; and to Messrs. J. S. Cushing & Co. for the typographical excellence of the book.

J. M. TAYLOR.

Hamilton, N. Y.,
Nov., 1884.

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ELEMENTS OF THE CALCULUS.

ELEMENTS OF THE CALCULUS.

CHAPTER I.

INTRODUCTION.

1. In the Calculus there are two kinds of quantities considered, *variables* and *constants*.

A Variable is a quantity that is, or is conceived to be, continually changing in value. Variables are usually represented by the final letters of the alphabet.

A Constant is a quantity whose value is fixed or invariable. Constants are usually represented by figures or the first letters of the alphabet. Particular values of variables are constants.

In the Calculus the locus of an equation is conceived as traced by a moving point called the **Generatrix**. If $a = \text{OB}$, the locus of $x^2+y^2=a^2$ is the circle ABCD. Now, as the generatrix traces this circle, its coördinates, x and y, continually change in value, and are therefore variables; while a retains the value OB, and is therefore a constant.

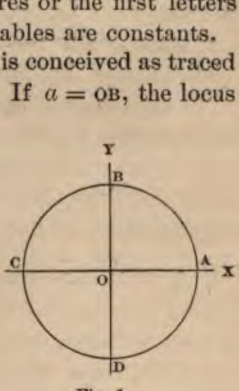

Fig. 1.

2. Functions and Independent Variables. One variable is a *function* of another, when the two are so related that any change of value in the second produces a change of value in the first.

For example, the area of a varying square is a function of its side; the volume of a variable sphere is a function of its radius; all mathematical expressions depending on x for their values, as ax^3, bx^4+cx^2, $\sin x$, $\log x$, etc., are functions of x.

An *independent variable* is one to which any arbitrary value or law of change may be assigned; as, x in x^3, x in $\sin x$, etc.

The symbol $f(x)$ is used to denote any function of x, and is read "function of x." When several functions of x occur in the same investigation, we employ other symbols, as $f'(x)$, $F(x)$, $\phi(x)$, etc., which are read "f prime function of x," "F function of x," "ϕ function of x," etc. According to this notation, $y=f(x)$ represents any equation between x and y, when solved for y.

3. Algebraic and Transcendental Functions. — An *algebraic* function is one that is expressed in terms of its variable or variables, by means of algebraic signs, without the use of variable exponents; as, ax^3-2cx^2, $5x^2-x$, etc.

All functions not algebraic are called *transcendental*. These are sub-divided into *exponential*, *logarithmic*, *trigonometric*, and *anti-trigonometric*.

An **Exponential** function is one in which the variable enters the exponent; as, a^{cx}, y^{ax}.

A **Logarithmic** function is one that involves the logarithm of a variable; as, $\log x$, $\log(bx+c)$.

The sine, cosine, tangent, etc., of a variable angle are called **Trigonometric** functions.

The symbol $\sin^{-1}x$, read "anti-sine of x," denotes the angle whose sine is x. $\text{Sin}^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, etc., are called **Inverse Trigonometric, or Anti-Trigonometric,** functions.

4. A variable is **Continuous**, or *varies continuously*, when, in passing from one value to another, it passes successively through all intermediate values.

A **Continuous** function is one that is constantly real, and varies continuously, when its variable varies continuously. Some functions are continuous for all real values of their variables, others only for those between certain limits. Thus, if $y=ax+b$, or $y=\sin x$, y is evidently a continuous function of x for all real values of x; but, if $y=\pm\sqrt{r^2-x^2}$, y is continuous only for values of x between the limits $-r$ and $+r$.

The Calculus treats of variables and functions only between their limits of continuity; hence all the values of x and $f(x)$ that it considers are represented geometrically by the coördinates of the points of the plane curve whose equation is $y=f(x)$.

Theory of Limits.

5. For convenience of reference, we give here a brief statement of the theory of limits.

The **Limit*** of a variable is a *constant quantity* which the variable, in accordance with its law of change, approaches indefinitely near, but which it never reaches. The variable may be less or greater than its limit.

Thus, if the number of sides of a regular polygon inscribed in or circumscribed about a circle be indefinitely increased, the area of the circle will be the *limit* of the area of either polygon, and the circumference will be the limit of the perimeter of either. When the polygons are inscribed, the variable area and perimeter are less than their limits; and, when the polygons are circumscribed, the variable area and perimeter are greater than their limits.

By increasing the number of terms, the sum of the series, $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} +$ etc., can be made to approach 2 as nearly as we please, but it cannot reach 2; hence 2 is the limit of the sum.

Again, if a point starting from A move the distance AC ($= \frac{1}{2}$ AB) the first second, the distance CD ($= \frac{1}{2}$ CB) the second second, and so on, AB will evidently be the limit of the line traced by this point.

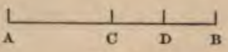

Fig. 2.

Cor. *The difference between a variable and its limit is a variable whose limit is zero.*

6. *If two variables are continually equal, and each approaches a limit, their limits are equal; that is, if* x = y, *and limit* (x) = a, *and limit* (y) = b, a = b.

For, since $x = y$, $a - x = a - y$; hence, as a is the limit of x, it is also of y (§ 5, Cor.). Since a and b each is a limit of y, and y cannot approach two unequal limits at the same time, $a = b$.

Cor. *If one of two continually equal variables approaches a limit, the other approaches the same limit.*

---

* The student should carefully note the two senses in which the word *limit* is used. In the theory of limits, a limit is a value which the variable cannot reach; in other cases, as in § 4, a limit is the greatest or the least value which the variable actually reaches.

7. *The limit of the product of a constant and a variable is the product of the constant and the limit of the variable; that is, if limit* (**x**) = **a**, *limit* (**cx**) = **ca.**

Let $v = a - x$;

then $cx = ca - cv$.

Now limit $(cv) = 0$, since limit $(v) = 0$;

hence limit $(cx) =$ limit $(ca - cv) = ca$.

8. *The limit of the variable product of two or more variables is the product of their limits; that is, if limit* (**x**) = **a**, *and limit* (**y**) = **b**, *limit* (**xy**) = **ab.**

Let $v = a - x$, and $v_1 = b - y$;

then $x = a - v$, and $y = b - v_1$;

$\therefore xy = ab - (av_1 + bv - vv_1)$.

Now limit $(av_1 + bv - vv_1)$* $= 0$;

hence limit $(xy) =$ limit $[ab - (av_1 + bv - vv_1)] = ab$.

In like manner, the theorem is proved for n variables.

9. *The limit of the variable quotient of two variables is the quotient of their limits; that is, if limit* (**x**) = **a**, *and limit* (**y**) = **b**, *limit* (**x** ÷ **y**) = (**a** ÷ **b**).†

Let $z = x \div y$,

and $c =$ limit (z), or limit $(x \div y)$.

Then $x = yz$; $\therefore a = bc$; §§ 6, 8.

$\therefore$ limit $(x \div y) [= c] = a \div b$.

10. *The limit of the variable sum of a finite number of variables is the sum of their limits; that is, if limit* (**x**) = **a**, *limit* (**y**) = **b**, *limit* (**z**) = **c**, *etc.*,

limit (**x** + **y** + **z** + ⋯) = **a** + **b** + **c** + ⋯

Let $v = a - x$, $v_1 = b - y$, $v_2 = c - z$, etc.

Then $x + y + z + \cdots$

$= (a + b + c + \cdots) - (v + v_1 + v_2 + \cdots)$;

$\therefore$ limit $(x + y + z + \cdots)$

$=$ limit $[(a + b + c + \cdots) - (v + v_1 + v_2 + \cdots)]$

$= a + b + c + \cdots$

---

* When v and v_1 have unlike signs, the difference, $av_1 + bv - vv_1$, may become zero for particular values of v and v_1, but it cannot remain zero, since xy is variable. The same is true of the difference, $v + v_1 + v_2 + \cdots$, in § 10.

† This principle does not hold when the limit of the divisor is zero.

COR. *When the product, quotient, or sum of two or more variables is equal to a constant, the product, quotient, or sum of their limits is equal to the same constant.*

11. The **Change** of a variable is **Uniform,** when its value changes equal amounts in equal arbitrary portions of time. In all other cases the change is variable. Thus, if from A toward B a point move equal distances, as Aa, ab, bc, etc., in equal arbitrary portions of time, the increase of the line traced will be uniform. Again, if the motion of a point along a straight line be uniform, the change of each of its rectilinear coördinates will evidently be uniform.

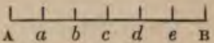

Fig. 3.

12. An **Increment** of a function or variable is the amount of its increase or decrease in any interval of time, and is found by subtracting its value at the beginning of the interval from its value at the end. Hence, if a variable is increasing, its increment is positive; and, if it is decreasing, its increment is negative. An increment of a variable is denoted by writing the letter Δ before it; thus, Δx, read "increment of x," is the symbol for an increment of x. If $y = f(x)$, Δx and Δy represent corresponding increments, that is, the increments of x and y in the same interval of time.

Let aPH be the locus of $y = f(x)$ referred to the rectangular axes OX and OY. If, when $x = \text{OA}$, $\Delta x = \text{OB} - \text{OA} = \text{AB}$; then $\Delta y = \text{BP}' - \text{AP} = \text{EP}'$; if, when $x = \text{OC}$, $\Delta x = \text{CF}$; then $\Delta y = \text{FH} - \text{CD} = -\text{ND}$.

In the last case Δy is negative, but it is properly called an increment, since it is what must be *added* to the first value to produce the second.

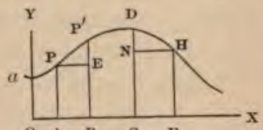

Fig. 4.

13. The **Differential** of a function or variable at any value is what *would* be its increment in any interval of time, if at that value its *change* became *uniform*. Hence, the differential of a variable is *positive* or *negative*, according as the variable is *increasing* or *decreasing*. The interval of time, though arbitrary, must be the same for a function as for its variable.

If the change of a variable be *uniform*, any *actual* increment may evidently be taken as its differential.

The differential of a variable is represented by writing the letter d before it; thus, dx, read "differential x," is the symbol for the differential of x. When the symbol of a function is not a single letter, parentheses are used; thus, $d(x^3)$ and $d(x^2-2x)$ denote the differentials of x^3 and x^2-2x.

14. Illustrations of Differentials. Conceive a variable right triangle as generated by the perpendicular moving uniformly to the right. Let y represent its area, x its base, and $2ax$ its altitude; then $y=ax^2$. Let BH be Δx estimated from the value AB$(=x')$, then BHMC will be Δy. But, if the increase of the area became uniform at the value ABC, the increment of the area in the same time would evidently be BHOC; hence, BHOC and BH may be taken as the differentials of y and x, when $x=x'$. But BHOC $=2ax'dx$, hence, in general, $dy[=d(ax^2)]=2axdx$. If $a=1$, $y=x^2$, and $dy=2xdx$. Here $\Delta y=dy+$ triangle COM.

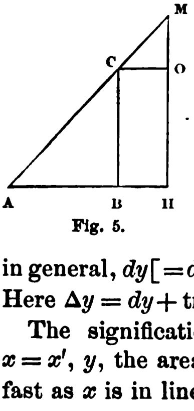

Fig. 5.

The signification of $dy=2ax'dx$ is evidently that, when $x=x'$, y, the area, is changing in units of surface $2ax'$ times as fast as x is in linear units.

Again, let OPn be the locus of $y=f(x)$, referred to the axes OX and OY. Conceive the area between OX and the curve as traced by the ordinate of the curve moving uniformly to the right. Let z represent this area, and let AB be Δx estimated from the value OA$(=x')$; then ABP′P $=\Delta z$. But, if the increase of z became uniform at the value OAP, its increment in the same interval would evidently be ABDP; hence AB and ABDP may be taken as the differentials of x and z respectively, when $x=x'$.

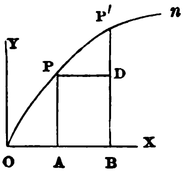

Fig. 6.

Hence $dz = \text{ABDP} = \text{AP}dx = y'dx$; or, in general, $dz = ydx$, which evidently means that z is changing y times as fast as x.

Area above the axis of x being positive, area below it is negative; hence, where the curve lies below the axis of x, the area decreases as x increases, and ydx is negative as it should be.

Here $\Delta z = dz +$ area PDP′.

15. The **Inclination** of a straight line referred to rectangular axes is the angle included between the axis of abscissas and the line. The *direction* of a line with respect to the axis of x is determined by its inclination.

The **Slope** of a line is the tangent of its inclination. Thus, in Fig. 7, HZP is the inclination of ZA, and tan HZP is the slope of ZA.

The direction of motion of the generatrix of a straight line is *constant*, while the direction of motion of the generatrix of a curve is *variable*.

A **Tangent** to a curve at any point is the straight line that passes through that point, and has the same direction as the curve at that point; or, a *tangent* to a curve at any point is the straight line that the generatrix *would* trace, if its direction of motion became constant at that point. The *slope* of a curve at any point is the slope of its tangent at that point. Thus, if, in Fig. 7, PA is a tangent to the curve at P, tan HZA is the slope of the curve at P.

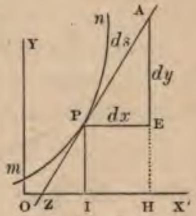

Fig. 7.

16. Geometric Signification of $\frac{dy}{dx}$. Let mn be the locus of $y = f(x)$, and let x' be the abscissa of any point upon it, as P. If at P the motion of the generatrix of the curve became uniform along the tangent PA, it is evident that the change of each of its coördinates would also become uniform. Hence PE, EA, and PA may be taken respectively as the differentials of x, y, and the length of the curve, when $x = x'$; for they are what would be the simultaneous increments of these variables, if the change of each became uniform at the value considered. Therefore $\frac{dy}{dx} = \frac{\text{EA}}{\text{PE}} = \tan \text{EPA}$

$= \tan$ HZA, which is the *slope* of the curve at P. Hence, in general, $\frac{dy}{dx}$ *is the slope of the curve* $y = f(x)$ *at any point* (x, y).

COR. 1. If EA, or dy, be c times as great as PE, or dx, y is evidently increasing c times as fast as x, when $x =$ OI.

COR. 2. If s represent the length of the curve mn, PA $= ds$, and $ds^2 = dx^2 + dy^2$, in which ds^2 denotes the square of ds.

17. Limit of the Ratio of the Increments of y and x.

Let mn be the locus of $y = f(x)$, and ED, a tangent at P, any point upon it; then the slope of this tangent $= \frac{dy}{dx}$ (§ 16). Let MN [= PC] $= \Delta x$, when estimated from the value OM, then CP′ $= \Delta y$.

Draw the secant PP′; then $\frac{\Delta y}{\Delta x}$ = the slope of the secant PP′. Conceive Δx to approach 0 as its limit; then the slope of the secant will approach the slope of the tangent as its limit.*

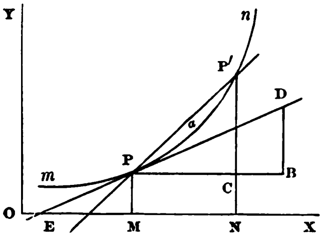

Fig. 8.

$$\therefore \lim_{\Delta x \doteq 0}\left[\frac{\Delta y}{\Delta x}\right]^{\dagger} = \frac{dy}{dx}.$$

Hence, *the ratio of the differential of a function to that of its variable is the limit of the ratio of their increments, as these increments approach zero as their limit.*

COR. 1. It is evident that $\frac{dy}{dx}$, or $\lim_{\Delta x \doteq 0}\left[\frac{\Delta y}{\Delta x}\right]$, is finite, except where the locus of $y = f(x)$ is parallel or perpendicular to the axis of x, where it is 0 or ∞.

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* This statement, if not sufficiently evident, may be demonstrated as follows: When the arc PaP′ is continuous in curvature, and this arc can always be made so small that it will be continuous, the slope of the secant PP′ is equal to the slope of a tangent to the arc PaP′ at some point, as a. Now, as the arc PaP′ approaches zero as its limit, the point a approaches P as its limiting position; hence the slope of the secant PP′ approaches the slope of the tangent PD as its limit.

† $\lim_{\Delta x \doteq 0}\left[\frac{\Delta y}{\Delta x}\right]$ is read "the limit of $\frac{\Delta y}{\Delta x}$ as Δx approaches 0 as its limit."

COR. 2. If $\frac{\Delta y}{\Delta x}$ be constant, the locus of $y=f(x)$ is evidently a straight line, in which case $\frac{\Delta y}{\Delta x}=\frac{dy}{dx}$.

COR. 3. A tangent to *mn* at P is evidently the limiting position of the secant PP′ as P′ approaches P and arc P′P $\doteq 0$.

The following is another proof of the important principle established above: —

SECOND PROOF.* Conceive the area between OX and the curve OP*n* (Fig. 9) as traced by the ordinate of the curve moving to the right. Let z represent this area, and let AB be Δx estimated from the value OA; then $\Delta y =$ DP′, and $\Delta z =$ ABP′P.

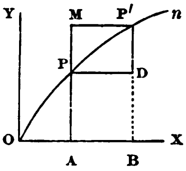

Fig. 9.

Now ABDP < ABP′P < ABP′M;

$$\therefore y'\Delta x < \Delta z < (y'+\Delta y)\Delta x.$$

Dividing by Δx, we have

$$y' < \frac{\Delta z}{\Delta x} < y' + \Delta y.$$

Whence $\frac{\Delta z}{\Delta x}$ differs from y' less than $y'+\Delta y$ does; but

$$\underset{\Delta x \doteq 0}{\text{limit}}\,[y'+\Delta y] = y'; \quad \therefore \underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\Delta z}{\Delta x}\right] = y'.$$

But $$\frac{dz}{dx} = y'. \qquad \S\ 14.$$

Hence $$\underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\Delta z}{\Delta x}\right] = \frac{dz}{dx}.$$

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* This demonstration assumes that any function of x may be represented graphically by the area between a curve and the axis of x. That many functions of x may be thus represented is very evident, and that any may be follows from § 67.

CHAPTER II.

DIFFERENTIATION.

18. Differentiation is the operation of finding the differential of a function. The sign of differentiation is d; thus d in $d(x^3)$ indicates the operation of differentiating x^3, while the whole expression $d(x^3)$ denotes the differential of x^3 (see § 13).

To *differentiate* ax^2, let $y = ax^2$, and let x' and y' be any corresponding values of x and y; then

$$y' = ax'^2. \tag{1}$$

Let Δx be any increment of x estimated from the value x', and Δy the corresponding increment of y; then

$$y' + \Delta y = a(x' + \Delta x)^2 = ax'^2 + 2ax'\Delta x + a(\Delta x)^2. \tag{2}$$

Subtracting (1) from (2), we have

$$\Delta y = 2ax'\Delta x + a(\Delta x)^2, \text{ or } \frac{\Delta y}{\Delta x} = 2ax' + a\Delta x. \tag{3}$$

$$\therefore \lim_{\Delta x \doteq 0}\left[\frac{\Delta y}{\Delta x}\right] = \lim_{\Delta x \doteq 0}\left[2ax' + a\Delta x\right]. \qquad \text{§ 6.}$$

$$\therefore \frac{dy}{dx} = 2ax', \text{ or, in general, } dy = 2axdx. \qquad \text{§ 17.}$$

By this general method we could differentiate any other function, but in practice it is more expedient to use the rules which we proceed to establish.

Algebraic Functions.

19. *The differential of the product of a constant and a variable is the product of the constant and the differential of the variable.*

We are to prove that $d(ay) = a dy$, in which y is some function of x. Let $u = ay$, and let x' represent any value of x, and y' and u' the corresponding values of y and u; then

$$u' = ay'. \qquad (1)$$

Let Δx represent any increment of x, estimated from the value x', and let Δy and Δu represent the corresponding increments of y and u; then

$$u' + \Delta u = a(y' + \Delta y) = ay' + a\Delta y. \qquad (2)$$

Subtracting (1) from (2), member from member, we have

$$\Delta u = a\Delta y.$$

$$\therefore \frac{\Delta u}{\Delta x} = a\frac{\Delta y}{\Delta x}.$$

$$\therefore \underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\Delta u}{\Delta x}\right] = \underset{\Delta x \doteq 0}{\text{limit}}\left[a\frac{\Delta y}{\Delta x}\right] = a \cdot \underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\Delta y}{\Delta x}\right]. \qquad \S\S\ 6,\ 7.$$

$$\therefore \frac{du}{dx} = a\frac{dy}{dx}. \qquad \S\ 17.$$

Hence, as x' is any value of x, we have in general, by multiplying both members by dx,

$$du\,[= d(ay)] = a dy.$$

Cor. $d\left(\frac{y}{a}\right) = d\left(\frac{1}{a}y\right) = \frac{dy}{a}.$

20. *The differential of a constant is zero.*

This is evident, since the increment of a constant in any interval of time is zero.

21. *The differential of a polynomial is the algebraic sum of the differentials of its several terms.*

We are to prove that $d(v + y - z + a) = dv + dy - dz$, in which v, y, and z are functions of x.

Let $u = v + y - z + a$, and let x' represent any value of x, and v', y', z', and u' the corresponding values of v, y, z, and u; then

$$u' = v' + y' - z' + a. \qquad (1)$$

Let Δx represent any increment of x, estimated from the value x', and Δv, Δy, Δz, and Δu the corresponding increments of v, y, z, and u; then

$$u' + \Delta u = v' + \Delta v + y' + \Delta y - (z' + \Delta z) + a. \qquad (2)$$

Subtracting (1) from (2) we have

$$\Delta u = \Delta v + \Delta y - \Delta z.$$

$$\therefore \frac{\Delta u}{\Delta x} = \frac{\Delta v}{\Delta x} + \frac{\Delta y}{\Delta x} - \frac{\Delta z}{\Delta x}.$$

$$\therefore \underset{\Delta x \doteq 0}{\text{limit}} \left[\frac{\Delta u}{\Delta x}\right] = \underset{\Delta x \doteq 0}{\text{limit}} \left[\frac{\Delta v}{\Delta x} + \frac{\Delta y}{\Delta x} - \frac{\Delta z}{\Delta x}\right]. \qquad \S\ 6.$$

$$\therefore \frac{du}{dx} = \frac{dv}{dx} + \frac{dy}{dx} - \frac{dz}{dx}. \qquad \S\S\ 10,\ 17.$$

Hence, as x' is any value of x, we have in general

$$du\,[= d\,(v + y - z + a)] = dv + dy - dz.$$

22. *The differential of the product of two variables is the first into the differential of the second, plus the second into the differential of the first.*

We are to prove that $d\,(yz) = ydz + zdy$, in which y and z are functions of x.

Let $u = yz$, and let x' represent any value of x, and y', z', and u' the corresponding values of y, z, and u; then

$$u' = y'z'. \qquad (1)$$

Let Δx represent any increment of x estimated from the value x', and Δy, Δz, and Δu the corresponding increments of y, z, and u; then

$$\begin{aligned} u' + \Delta u &= (y' + \Delta y)(z' + \Delta z) \\ &= y'z' + y'\Delta z + z'\Delta y + \Delta z \Delta y. \qquad (2) \end{aligned}$$

Subtracting (1) from (2) we have

$$\Delta u = y'\Delta z + z'\Delta y + \Delta z \Delta y.$$

$$\therefore \frac{\Delta u}{\Delta x} = y' \frac{\Delta z}{\Delta x} + (z' + \Delta z)\frac{\Delta y}{\Delta x}.$$

$$\therefore \lim_{\Delta x \doteq 0}\left[\frac{\Delta u}{\Delta x}\right] = \lim_{\Delta x \doteq 0}\left[y'\frac{\Delta z}{\Delta x}\right] + \lim_{\Delta x \doteq 0}\left[(z' + \Delta z)\frac{\Delta y}{\Delta x}\right].$$

$$\therefore \frac{du}{dx} = y'\frac{dz}{dx} + z'\frac{dy}{dx}. \qquad \S\S\ 7,\ 8.$$

Hence, as x' is any value of x, we have in general

$$du\,[= d(yz)] = ydz + zdy.$$

To obtain this result geometrically, let z and y represent the variable altitude and base of a rectangle conceived as generated by the side z moving to the right, and the upper base y moving upward; then $zy =$ its area.

If, at the value DCBA (Fig. 10), $dz =$ AH, and $dy =$ CE, d(area) $=$ CEFB + BGHA; since CEFB + BGHA is evidently what would be the increment of the area of the rectangle in the assumed interval, if at the value DCBA the increase of its area became uniform.

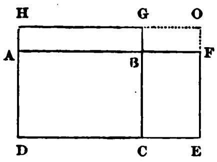

Fig. 10.

Hence, $d(zy) = d(\text{area}) =$ CEFB + BGHA $= zdy + ydz$.

Here $\quad \Delta(zy) = d(zy) +$ BGOF.

23. *The differential of the product of any number of variables is the sum of the products of the differential of each into all the rest.*

We are to prove that $d(xyz) = yzdx + xzdy + xydz$, in which y and z are functions of x.

Let $u = xy$, then $d(xyz) = d(uz)$.

$$\begin{aligned} \text{But} \quad d(uz) &= zdu + udz \qquad \S\ 22. \\ &= zd(xy) + xydz \\ &= yzdx + xzdy + xydz. \end{aligned}$$

$$\therefore d(xyz) = yzdx + xzdy + xydz.$$

In a similar manner, the theorem may be demonstrated for any number of variables.

24. *The differential of a fraction is the denominator into the differential of the numerator, minus the numerator into the differential of the denominator, divided by the square of the denominator.*

We are to prove that $d\left(\frac{y}{z}\right)=\frac{zdy-ydz}{z^2}$, in which y and z are functions of x.

Let $u=\frac{y}{z}$, then $uz=y$.

$$\therefore udz+zdu=dy.$$

$$\therefore du=\frac{dy-udz}{z}=\frac{dy-\frac{y}{z}dz}{z}=\frac{zdy-ydz}{z^2}.$$

Cor. $d\left(\frac{a}{x}\right)=\frac{xda-adx}{x^2}=-\frac{adx}{x^2}$, since $da=0$; that is, *the differential of a fraction with a constant numerator is minus the numerator into the differential of the denominator divided by the square of the denominator.*

25. *The differential of a variable affected with any constant exponent is the product of the exponent, the variable with its exponent diminished by one, and the differential of the variable.*

I. *When the exponent is a positive integer.*

If n is a positive integer, $x^n=x\cdot x\cdot x$ to n factors; hence we have

$$\begin{aligned} d(x^n) &= d(x\cdot x\cdot x \text{ to } n \text{ factors})\\ &= x^{n-1}dx+x^{n-1}dx+\text{etc. to } n \text{ terms} \qquad \S\ 23.\\ &= nx^{n-1}dx. \end{aligned}$$

II. *When the exponent is a positive fraction.*

Let $y=x^{\frac{m}{n}}$, then $y^n=x^m$. (1)

Differentiating (1), we obtain

$$ny^{n-1}dy=mx^{m-1}dx.$$

$$\therefore dy = \frac{m}{n}\frac{x^{m-1}}{y^{n-1}}\,dx = \frac{m}{n}\frac{x^{m-1}y}{y^n}\,dx = \frac{m}{n}\frac{x^{m-1}x^{\frac{m}{n}}}{x^m}\,dx$$

$$= \frac{m}{n}x^{\frac{m}{n}-1}\,dx.$$

III. *When the exponent is negative.*

Let $y = x^{-n}$, n being integral or fractional; then

$$y = \frac{1}{x^n}. \qquad (1)$$

Differentiating (1), we have

$$dy = -\frac{nx^{n-1}}{x^{2n}}dx = -nx^{-n-1}dx. \qquad \text{§ 24, Cor.}$$

For a proof of this theorem, which includes the case of incommensurable exponents, see § 39, Ex. 25.

Assuming the binomial theorem, let the student prove this rule by the general method of differentiation.

COR. $d(\sqrt{x}) = \frac{1}{2}x^{-\frac{1}{2}}dx = \frac{dx}{2\sqrt{x}}.$

26. The general symbol for the differential of $f(x)$ is $f'(x)dx$; hence, if $y = f(x)$, $dy = f'(x)dx$.

EXAMPLES.

Differentiate

1. $x^3 + 8x + 2x^2$. *Ans.* $(3x^2 + 8 + 4x)\,dx$.

$d(x^3 + 8x + 2x^2) = d(x^3) + d(8x) + d(2x^2)$, § 21.

$d(x^3) = 3x^2dx$, § 25.

$d(8x) = 8dx$, § 19.

$d(2x^2) = 4xdx$, §§ 19, 25.

$\therefore d(x^3 + 8x + 2x^2) = (3x^2 + 8 + 4x)\,dx.$

2. $y = 3ax^2 - 5nx - 8m$. $dy = (6ax - 5n)\,dx$.

$dy = d(3ax^2 - 5nx - 8m) = d(3ax^2) - d(5nx) - d(8m).$

[The differentials of equals are equal, and § 21.]

3. $f(x) = 5ax^2 - 3b^2x^3 - abx^4.$

$$f'(x)\,dx = (10ax - 9b^2x^2 - 4abx^3)\,dx.$$

4. $f(x) = a^3 + 5b^2x^3 + 7a^3x^5.$

$$f'(x)\,dx = (15b^2x^2 + 35a^3x^4)\,dx.$$

5. $y = ax^{\frac{3}{2}} + bx^{\frac{1}{2}} + c.$ $\qquad dy = \dfrac{3ax + b}{2\sqrt{x}}dx.$

6. $y^2 = 2px.$ $\qquad \dfrac{dy}{dx} = \dfrac{p}{y}.$

$d(y^2) = d(2px).$

7. $a^2y^2 + b^2x^2 = a^2b^2.$ $\qquad dy = -\dfrac{b^2x}{a^2y}dx.$

8. $f(x) = (b + ax^2)^{\frac{5}{4}}.$ $\qquad f'(x)dx = \frac{5}{2}(b + ax^2)^{\frac{1}{4}}axdx.$

9. $y = (1 + 2x^2)(1 + 4x^3).$ $\qquad dy = 4x(1 + 3x + 10x^3)dx.$

$dy = (1 + 2x^2)d(1 + 4x^3) + (1 + 4x^3)d(1 + 2x^2).$

10. $y = \dfrac{x + a^2}{x + b}.$ $\qquad dy = \dfrac{b - a^2}{(x + b)^2}dx.$

$$dy = \frac{(x + b)d(x + a^2) - (x + a^2)d(x + b)}{(x + b)^2}.$$

11. $f(x) = \dfrac{a}{b - 2x^2}.$ $\qquad f'(x)dx = \dfrac{4ax}{(b - 2x^2)^2}dx.$

12. $y = (a + x)\sqrt{a - x}.$ $\qquad dy = \dfrac{a - 3x}{2\sqrt{a - x}}dx.$

13. $f(x) = \dfrac{2x^4}{a^2 - x^2}.$ $\qquad f'(x)dx = \dfrac{8a^2x^3 - 4x^5}{(a^2 - x^2)^2}dx.$

14. $f(x) = \left(\dfrac{x}{1 - x}\right)^m.$ $\qquad f'(x)dx = \dfrac{mx^{m-1}}{(1 - x)^{m+1}}dx.$

15. $f(x) = \sqrt{\dfrac{1 + x}{1 - x}}.$ $\qquad f'(x) = \dfrac{1}{(1 - x)\sqrt{1 - x^2}}.$

16. $f(x) = \dfrac{x^3}{(1 - x^2)^{\frac{3}{2}}}.$ $\qquad f'(x) = \dfrac{3x^2}{(1 - x^2)^{\frac{5}{2}}}.$

17. $y = \dfrac{x}{\sqrt{1 + x^2}}.$

18. $2xy^2 - ay^2 = x^3.$ $\qquad dy = \frac{3x^2 - 2y^2}{4xy - 2ay}dx.$

19. $f(x) = \frac{2x^2 - 3}{4x + x^2}.$ $\qquad f'(x)dx = \frac{8x^2 + 6x + 12}{(4x + x^2)^2}dx.$

20. $y = \frac{x^3}{a^2 - x^2}.$

21. $f(x) = \sqrt{ax} + \sqrt{c^2x^3}.$ $\qquad f'(x) = \frac{\sqrt{a} + 3cx}{2\sqrt{x}}.$

22. $\frac{a}{(b^2 + x^2)^3}.$ $\qquad -\frac{6ax}{(b^2 + x^2)^4}dx.$

23. $f(x) = \frac{x^3}{(1 + x)^2}.$ $\qquad f'(x) = \frac{3x^2 + x^3}{(1 + x)^3}.$

24. $f(x) = \frac{2x^2 - 1}{x\sqrt{1 + x^2}}.$ $\qquad f'(x) = \frac{1 + 4x^2}{x^2(1 + x^2)^{\frac{3}{2}}}.$

25. $(x - 2y)(b - 3x) = (c - x^2)(1 - y).$

$$\frac{dy}{dx} = \frac{b - 4x + 6y - 2xy}{x^2 - 6x - c + 2b}.$$

27. An **Increasing** function is one that increases when its variable increases; hence it decreases when its variable decreases.

A **Decreasing** function is one that decreases when its variable increases; hence it increases when its variable decreases. Thus, ax and a^x are increasing, and $\frac{1}{x}$ and $a - x$ are decreasing functions of x.

28. The **Derivative** of a function is the ratio of the differential of the function to the differential of its variable. This ratio is sometimes called the *derived function* or the *differential coefficient.* Hence the derivative of $f(x)$ is $f'(x)$, or the ratio of $f'(x)dx$ to dx. The derivative of y with respect to x is represented by $\frac{dy}{dx}$. If $y = f(x)$, then $\frac{dy}{dx} = f'(x)$. Thus, if $y = x^3$, $\frac{dy}{dx} = 3x^2$; that is, $3x^2$ is the derivative of y, or x^3; if $f(x) = x^6$, then $f'(x) = 6x^5$.

29. The **Measure** of the *rate* of *change* of a variable at a given instant is what *would* be its increment in a *unit* of time, if at that instant its *change* became *uniform*. This measure of rate is generally called *the rate*. Hence, the rate will be positive or negative, according as the variable is increasing or decreasing. Thus, when we say that the distance of a train from the station was changing, at a given instant, at the rate of +30 miles an hour, we mean that this distance *would have increased* thirty miles in an hour, if at that instant its increase had become uniform.

If the change of a variable is uniform, the actual increment of the variable in a unit of time is the measure of its rate.

30. Signification of $\frac{dy}{dt}$. Let t represent time; then, any variable, as y, is evidently some function of t. Since time changes uniformly, dt may represent any increment or interval of time. If dt equals the unit of time, then by definition dy equals the measure of the rate of change of y; and, if dt is n times the unit of time, dy is n times the rate of change of y; hence, *whatever be the value of* dt, $\frac{\mathrm{dy}}{\mathrm{dt}}$ = *the rate of change of* y.

31. Signification of $\frac{dy}{dx}$, or $f'(x)$. $\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}=$ the ratio of the rate of change of y to that of x (§ 30). $f'(x)=\frac{f'(x)dx}{dt}\div\frac{dx}{dt}$ = the ratio of the rate of change of $f(x)$ to that of x.

Hence, *the derivative of a function expresses the ratio of the rate of change of the function to that of its variable; and a function is an increasing function or a decreasing function, according as its derivative is positive or negative.*

COR. The same function of x may be an increasing function for some values of x, and a decreasing one for other values. Thus, since $-\frac{2a}{x^3}$, the derivative of $\frac{a}{x^2}$, is + when $x<0$, and − when $x>0$, $\frac{a}{x^2}$ is an increasing function when $x<0$, and a decreasing function when $x>0$.

32. When the change of y is uniform, it is evident that $\frac{\Delta y}{\Delta t}$ is the rate of change of y. When the change of y is variable, the value of $\frac{\Delta y}{\Delta t}$ evidently lies between the greatest and the least values of the rate of change of y during the time Δt; hence, the smaller Δt is taken, the nearer $\frac{\Delta y}{\Delta t}$ approaches the rate of change of y at the beginning of Δt.

Hence,
$$\underset{\Delta t \doteq 0}{\text{limit}}\left[\frac{\Delta y}{\Delta t}\right] = \left\{\begin{array}{l}\text{the rate of change of } y\\ \text{at the beginning of } \Delta t\end{array}\right\} = \frac{dy}{dt}; \quad (1)$$

and
$$\underset{\Delta t \doteq 0}{\text{limit}}\left[\frac{\Delta x}{\Delta t}\right] = \left\{\begin{array}{l}\text{the rate of change of } x\\ \text{at the beginning of } \Delta t\end{array}\right\} = \frac{dx}{dt}. \quad (2)$$

Dividing (1) by (2), we obtain, without the aid of a locus,

$$\underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\Delta y}{\Delta x}\right] = \frac{dy}{dx}. \qquad \S\ 9.$$

Applications.

1. The area of a circular plate of metal expanded by heat increases how many times as fast as its radius? If, when the radius is two inches, it is increasing at the rate of .01 inch a second, how fast is the area increasing at the same time?

Let $x =$ the radius, and y the area of the plate; then $y = \pi x^2$. $\therefore\ dy = 2\pi x dx$, or $\frac{dy}{dt} = 2\pi x \frac{dx}{dt}$; that is, the area is increasing in square inches $2\pi x$ times as fast as the radius is in linear inches. When $x = 2$, and $\frac{dx}{dt} = .01$ in., $\frac{dy}{dt} = .04\pi$ sq. in.; that is, the area is increasing $.04\pi$ sq. in. a second at the instant considered.

2. The volume of a spherical soap-bubble increases how many times as fast as its radius? When its radius is 3 in., and is increasing at the rate of 2 in. a second, how fast is its volume increasing?

Ans. The volume is increasing in cubic inches $4\pi x^2$ times as fast as the radius is in linear inches. The volume is increasing 72π cu. in. a second at the instant considered.

3. A boy is running on a horizontal plane in a straight line towards the base of a tower 50 metres in height. He is approaching the top how many times as fast as he is the foot of the tower? How fast is he approaching the top, when he is 500 metres from the foot, and running at the rate of 200 metres a minute?

Let x and y respectively represent in metres the distances of the boy from the foot and the top of the tower; then $y^2 = x^2 + (50)^2$, etc. *Ans.* 199 metres a minute.

4. A light is 4 metres above and directly over a straight horizontal side-walk, on which a man $1\frac{2}{3}$ metres in height is walking away from the light. The farthest point of the man's shadow is moving how many times as fast as he is walking? The man's shadow is lengthening how many times as fast as he is walking? How fast is the shadow lengthening, and its farthest point moving, when the man is walking at the rate of 50 metres a minute?

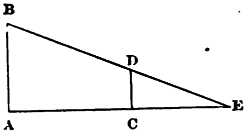

Fig. 11.

Let AE be the sidewalk, B the position of the light, and CD one position of the man. Let AE $= y$, and AC $= x$; then $y - x : y :: \frac{5}{3} : 4$; $\therefore dy = \frac{12}{7}\,dx$. Again, let $y =$ CE, and $x =$ AC; then $y + x : y :: 4 : \frac{5}{3}$; $\therefore dy = \frac{5}{7}\,dx$.

5. The altitude of a variable cylinder is constantly equal to the diameter of its base. In general, its volume is changing how many times as fast as its altitude? If, when its altitude is 6 metres, it is increasing at the rate of 2 metres an hour, how fast is its volume increasing at the same instant? How fast is the entire surface increasing at the same instant?

Ans. $\frac{3}{4}\pi x^2$ times, x being its altitude; 54π kilolitres an hour; 36π centiares an hour.

6. The altitude of a varying frustum of a right cone is constantly equal to the radius of its lower base, and the radius of its upper base is one-half that of its lower base. If, when the radius of its lower base is 4 metres, it is increasing at the rate of 2 metres an hour, how fast is the volume of the frustum increasing at the same instant?

7. The area of an equilateral triangle increases how many times as fast as each of its sides? How fast is its area increasing when each of its sides is 10 in., and increasing at the rate of 3 in. a second? What is the length of each of its sides, when its area is increasing in square inches 30 times as fast as each of its sides is in linear inches?

Ans. $15\sqrt{3}$ sq. in. a second · $20\sqrt{3}$ in.

8. One end of a ladder 20 ft. long was on the ground 5 ft. from the foundation of a building, which stood on a horizontal plane, while the other end rested against the side of the building. The end on the ground was carried away from the building on a line perpendicular to it, at the uniform rate of 4 ft. a minute; how fast did the other end begin to descend along the building? How fast was it descending at the end of two minutes? How far was the foot of the ladder from the building, when the top was descending at the rate of 4 ft. a minute?

Ans. 1.03+ ft. a minute; 3.42 ft. a minute; $10\sqrt{2}$ ft.

9. In the parabola whose parameter is 8, the ordinate changes how many times as fast as the abscissa? What is its slope at any point (x, y)? Find its inclination at the points whose common abscissa is $\frac{1}{2}$. Is y an increasing or a decreasing function of x? At what points does the ordinate change numerically four times as fast as the abscissa?

In this case, y is a *two-valued* function; and $\frac{4}{y}$ is + or −, according as y is + or −; ∴ the + value of y is an increasing, and the − value a decreasing, function of x.

Ans. $\frac{4}{y}$; $\frac{4}{y}$; 63° 26′ 6″ and 116° 33′ 54″; $(\frac{1}{8}.\ 1)$ and $(\frac{1}{8}, -1)$.

10. In the ellipse $a^2y^2 + b^2x^2 = a^2b^2$, the ordinate increases how many times as fast as the abscissa? y changes how many times as fast as x at the extremities of the axes of the curve? How can the points be found at which y changes c times as fast as x? What is the slope of the ellipse at any point? What, at the extremities of its axes? Is y an increasing or a decreasing function of x?

$\frac{dy}{dx} = -\frac{b^2x}{a^2y}$; $\therefore$ when y changes c times as fast as x, $-\frac{b^2x}{a^2y} = c$. When x and y have unlike signs, $-\frac{b^2x}{a^2y}$ is $+$, and y is an increasing function; when x and y have like signs, $-\frac{b^2x}{a^2y}$ is $-$, and y is a decreasing function.

11. What is the slope of $y^2 = x^3 + 2x^4$ at (x, y)? What is it for $x = 2$?

Ans. $\pm\frac{3x + 8x^2}{2\sqrt{x + 2x^2}}$; $\pm\frac{19}{10}\sqrt{10}$.

12. What is the slope of $y = x^3 - x^2 + 1$ at the point whose abscissa is 2? 1? 0? -1? *Ans.* $+8$; $+1$; $+0$; $+5$.

13. At what point on $y^2 = 2x^3$ is the slope 3? At what point is the curve parallel to the axis of x? *Ans.* (2, 4); (0, 0).

14. At what angles does the line $3y - 2x - 8 = 0$ cut the parabola $y^2 = 8x$?

Find their slopes at their points of intersection; then find the angles between the lines having these slopes.

Ans. $\tan^{-1}.2$ and $\tan^{-1}.125$.

15. One ship was sailing south at the rate of 6 miles an hour; another, east at the rate of 8 miles an hour. At 4 P.M. the second crossed the track of the first at a point where the first was two hours before. How was the distance between the ships changing at 3 P.M.? How at 5 P.M.? When was the distance between them not changing?

Let t = the time in hours, reckoned from 4 P.M., time after 4 P.M. being $+$, and time before $-$. Then $8t$ and $6t + 12$ will represent respectively the distances of the two ships from the point of intersection of their paths, distances south and east being $+$, and distances west and north being $-$. Let y = the distance between the ships; then,

$$y^2 = (8t)^2 + (6t + 12)^2.$$

$$\therefore \frac{dy}{dt} = \frac{100t + 72}{[64t^2 + (6t + 12)^2]^{\frac{1}{2}}}. \qquad (1)$$

$3x^{2}-$

When y does not change, $dy = 0$. $\therefore$ from (1), $100t + 72 = 0$; $\therefore$ $t = -.72$ of 60 minutes $= -43.2$ minutes. Therefore, the distance between them was not changing at 43.2 minutes before 4 P.M., or at 16.8 minutes after 3 P.M.

Ans. Diminishing 2.8 miles an hour; increasing 8.73.

33. Velocity is the rate of change of the distance passed over by a moving body. Hence, if $s =$ the distance and $v =$ the velocity, $v = \frac{ds}{dt}$ (§ 30). If the unit of s is one foot, and the unit of t one second, $v = \frac{ds}{dt}$ ft. a second.

Acceleration is the rate of change of velocity.

Hence, if $a =$ acceleration, $a = \frac{dv}{dt}$ (§ 30).

EXAMPLES.

1. If $s = 2t^3$, what is the velocity and acceleration?

Here $v = \frac{ds}{dt} = 6t^2$ ft. a second; $a = \frac{dv}{dt} = 12t$ ft. a second; and the rate of change of acceleration $= \frac{da}{dt} = 12$ ft. a second.

2. If $g = 32.17$ ft., $s = \frac{g}{2}t^2$ is the law of falling bodies in vacuo near the earth's surface; find the velocity and acceleration in general, also at the end of the third and the eighth second.

Ans. $a = 32.17$ ft. a second; $v = 32.17t$ ft. a second; 96.51; 257.36.

3. Given $s = at^{\frac{1}{2}}$, to find v and a in general, and at the end of 4 seconds.

Ans. $v = \frac{a}{2\sqrt{t}}$ and $\frac{a}{4}$ ft. a second; $a = -\frac{a}{4\sqrt{t^3}}$ and $-\frac{a}{32}$ ft. a second; that is, the velocity decreases at the rate of $\frac{a}{4\sqrt{t^3}}$ ft. a second.

4. Given $s^3 = 8t^2$, to find v and a in general, and at the end of 8 seconds.

Ans. $v = \dfrac{4}{3\sqrt[3]{t}}$ and $\frac{2}{3}$ ft. a second.

5. A point moves along a parabola with a velocity v'; required the rates of change of its coördinates.

$$\text{Since } y^2 = 2px,\ \frac{dy}{dx} = \frac{p}{y}. \tag{1}$$

If s represents the length of the curve traversed, by the conditions of the problem, we have

$$\frac{ds}{dt} = v'. \tag{2}$$

$$\text{But}\quad \frac{ds}{dt} = \frac{dy}{dt} \cdot \frac{ds}{dy} = \frac{dy}{dt} \cdot \frac{\sqrt{dx^2 + dy^2}}{dy} = \frac{dy}{dt}\sqrt{1 + \frac{dx^2}{dy^2}}, \tag{3}$$

since $ds = \sqrt{dx^2 + dy^2}$. § 16, Cor. 2.

From (1), (2), and (3), we obtain

$$v' = \frac{dy}{dt}\sqrt{1 + \frac{y^2}{p^2}}, \text{ or } \frac{dy}{dt} = \frac{p}{\sqrt{p^2 + y^2}}v',$$

which is the rate of change of y.

In like manner, we obtain

$$\frac{dx}{dt} = \frac{y}{\sqrt{p^2 + y^2}}v', \text{ the rate of change of } x.$$

Logarithmic and Exponential Functions.

34. *The differential of the logarithm of a variable is the quotient of the differential of the variable divided by the variable itself, multiplied by a constant.*

Let $y = nx;$ (1)

then $dy = ndx,$ (2)

and $\log_a y^* = \log_a n + \log_a x.$ (3)

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* $\log_a y$ is read, "the logarithm of y to the base a."

From (1) and (2),

$$\frac{dy}{y}=\frac{dx}{x}. \tag{4}$$

From (3),

$$d(\log_a y)=d(\log_a x). \tag{5}$$

From (5) and (4),

$$\frac{d(\log_a y)}{\frac{dy}{y}}=\frac{d(\log_a x)}{\frac{dx}{x}}.$$

Whence $d(\log_a x)$ bears the same ratio to $\frac{dx}{x}$ that $d(\log_a y)$ does to $\frac{dy}{y}$. Let m be this ratio for some particular value of x, as x'; then $d(\log_a x)=m\frac{dx}{x}$ when $x=x'$, and $d(\log_a y)=m\frac{dy}{y}$ when $y=nx'$; but, as n is an arbitrary constant, nx' may be any number. Hence, in general, $d(\log_a y)=m\frac{dy}{y}$, in which m is a constant.*

The constant m is called the **Modulus** of the system of logarithms whose base is a.

35. Let m and m' be the moduli of two systems of logarithms whose bases are a and b respectively. If $a > b$, it is evident that $\log_a x$ must change more slowly than $\log_b x$; $\therefore d(\log_a x) < d(\log_b x)$; that is,

$$m\frac{dx}{x}<m'\frac{dx}{x}, \text{ or } m < m'.$$

Hence, *the greater the base of a system of logarithms, the smaller is its modulus.*

36. Naperian System. The system of logarithms whose modulus is *unity* is called the *Naperian system.* The symbol for the Naperian *base* is e.

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* See Rice and Johnson's Calculus, p. 39; Olney's Calculus, p. 25; also Bowser's Calculus, p. 29.

The differential of the Naperian logarithm of a variable is, therefore, the differential of the variable divided by the variable.

Thus we see that Naperian logarithms are the simplest and most natural for analytic purposes; and, hereafter, the symbol *log* will stand for the Naperian logarithm.

37. *The differential of an exponential function with a constant base is equal to the function itself into the logarithm of the base into the differential of the exponent, divided by the modulus of the system of logarithms used.*

Let $y = c^x$, then $\log_a y = x \log_a c$;

$$\therefore m\frac{dy}{y} = \log_a c\,dx; \quad \therefore dy[=d(c^x)] = \frac{c^x \log_a c}{m}dx.$$

COR. In the Naperian system, the modulus being unity, we have

$$d(c^x) = c^x \log c\,dx;$$

also, $$d(e^x) = e^x dx, \text{ since } \log e = 1.$$

38. *The differential of an exponential function with a variable base is the sum of the results obtained by first differentiating as though the base were constant, and then as though the exponent were constant.*

Let $u = y^x$, then $\log u = x \log y$;

$$\therefore \frac{du}{u} = \log y\,dx + x\frac{dy}{y};$$

$$\therefore du = y^x \log y\,dx + xy^{x-1}dy,$$

which is the result obtained by following the rule given.

39. Logarithmic Differentiation. Exponential functions, as also those involving products and quotients, are often more easily differentiated by first passing to logarithms. This method, which is illustrated in the two preceding demonstrations, is called *logarithmic differentiation.*

EXAMPLES.

1. $y = \log(x^2 + x)$. $\qquad dy = \dfrac{2x+1}{x^2+x}dx.$

2. $y = \log_a \sqrt{1 - x^3}$. $\qquad dy = \dfrac{3mx^2}{2(x^3-1)}dx.$

3. $f(x) = \log_a x^3$. $\qquad f'(x) = \dfrac{3m}{x}.$

4. $f(x) = x\log x$. $\qquad f'(x) = \log x + 1.$

5. $y = c^{\log ax}$. $\qquad dy = \dfrac{c^{\log ax}\log c}{x}dx.$

6. $f(x) = (\log x)^3$. $\qquad f'(x)dx = \dfrac{3(\log x)^2}{x}dx.$

7. $f(x) = x^x$. $\qquad f'(x) = x^x(\log x + 1).$

8. $y = x^{x^x}$.

$$\log y = x^x \log x;\ \therefore \frac{dy}{y} = x^x\frac{dx}{x} + \log x[x^x(\log x + 1)]dx.$$

$$\therefore dy = x^{x^x}x^x\left[\log x(\log x + 1) + \frac{1}{x}\right]dx.$$

9. $y = x^{e^x}$. $\qquad \dfrac{dy}{dx} = x^{e^x}e^x\dfrac{1 + x\log x}{x}.$

10. $y = \dfrac{a^x - 1}{a^x + 1}$. $\qquad dy = \dfrac{2a^x \log a dx}{(a^x+1)^2}.$

11. $y = \log\dfrac{x}{\sqrt{1+x^2}}$. $\qquad \dfrac{dy}{dx} = \dfrac{1}{x(1+x^2)}.$

12. $y = \log(\log x)$. $\qquad \dfrac{dy}{dx} = \dfrac{1}{x\log x}.$

13. $y = \dfrac{c^x}{x^x}$. $\qquad \dfrac{dy}{dx} = \left(\dfrac{c}{x}\right)^x\left(\log\dfrac{c}{x} - 1\right).$

14. $y = \dfrac{\sqrt{1+x}}{\sqrt{1-x}}$. $\qquad dy = \dfrac{dx}{(1-x)\sqrt{1-x^2}}.$

In this example and some that follow, pass to logarithms.

15. $y = \frac{x^n}{(a+x)^n}.$ $\qquad \frac{dy}{dx} = \frac{nax^{n-1}}{(a+x)^{n+1}}.$

16. $y = \frac{(a^2+x^2)^{\frac{3}{2}}}{(a^2-x^2)^{\frac{1}{2}}}.$ $\qquad \frac{dy}{dx} = 2x\frac{2a^2-x^2}{(a^2-x^2)^{\frac{3}{2}}}(a^2+x^2)^{\frac{1}{2}}.$

17. $y = \log\frac{e^x}{1+e^x}.$ $\qquad \frac{dy}{dx} = \frac{1}{1+e^x}.$

18. $y = x^{\frac{1}{x}}.$ $\qquad \frac{dy}{dx} = \frac{x^{\frac{1}{x}}(1-\log x)}{x^2}.$

19. $y = e^x(1-x^3).$ $\qquad \frac{dy}{dx} = e^x(1-3x^2-x^3).$

20. $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$ $\qquad \frac{dy}{dx} = \frac{4}{(e^x+e^{-x})^2}.$

21. $y = \left(\frac{x}{n}\right)^{nx}.$ $\qquad \frac{dy}{dx} = n\left(\frac{x}{n}\right)^{nx}\left(1+\log\frac{x}{n}\right).$

22. $y = (a^x+1)^2.$ $\qquad dy = 2a^x(a^x+1)\log a\,dx.$

23. Which increases the more rapidly, a number or its logarithm?

Let $y = \log_a x$, then $dy = \frac{m}{x}dx$; hence $\log_a x$ changes faster or more slowly than x, according as $x <$ or $> m$.

Since, in the Naperian system, $m = 1$, $dx = x\,dy$; that is, the number x changes x times as fast as $\log_e x$.

Rem. The ratio of the rate of change of a number to that of its logarithm is variable; and yet the hypothesis, that it is constant for comparatively small changes in the number, is sufficiently accurate for practical purposes, and is the assumption made in using the tabular differences in tables of logarithms.

24. If $y = \log_{10}x$, x changes how many times as fast as y, when $x = 2560$, the modulus of the Common system being .434294?

$$\frac{dx}{dy} = \frac{x}{m} = \frac{2560}{.434294} = 5895 \text{ nearly.}$$

25. By means of the formula $d(\log x) = \frac{dx}{x}$, find $d(x^n)$ in which n is any number, commensurable or incommensurable.

Let $u = x^n$, then $\log u = n \log x$;

$$\therefore \frac{du}{u} = n\frac{dx}{x}, \text{ or } du = nx^{n-1}dx.$$

If x were negative, to avoid logarithms of negative numbers, we would square both members of $u = x^n$ before differentiating.

26. In like manner, obtain $d(xy)$, $d(xyz)$, and $d\left(\frac{x}{y}\right)$.

27. Prove that $d\left(\frac{1}{2a}\log\frac{x-a}{x+a} + \log c\right) = \frac{dx}{x^2 - a^2}$.

28. Prove that $d\left[\log(x + \sqrt{x^2 \pm a^2}) + \log c\right] = \frac{dx}{\sqrt{x^2 \pm a^2}}$.

29. What is the slope of the curve $x = \log_{10} y$, or $y = 10^x$? What at $x = 0$? What at $y = 5$?

Ans. $\frac{y}{.43429}$; 2.3+ ; 11.51+.

Trigonometric Functions.

40. In the higher mathematics, the *unit* of *angular measure* is the angle whose measuring arc is a radius in length; hence, if x represents the length of the measuring arc of any angle, and r its radius, the angle equals $\frac{x}{r}$; or, if $r = 1$, the angle $= x$. In what follows we shall assume $r = 1$.

41. *The differential of the sine of an angle is equal to the cosine of the angle into the differential of the angle.*

The differential of the cosine of an angle is equal to minus the sine of the angle into the differential of the angle.

For, let x represent any angle, or its measuring arc, and let AC be any value of this arc. If at C the motion of the generatrix became uniform along the tangent CD, it is evident that any simultaneous increments of its distances from C and lines BA and BH may be taken as the differentials of the arc, the sine, and the cosine, when $x = \text{AC}$; that is, if $\text{CD} = dx$, $\text{ED} = d(\sin x)$, and $-\text{EC} = d(\cos x)$. Now angle $\text{EDC} = \text{ABC} = x$; $\therefore$ in triangle EDC,

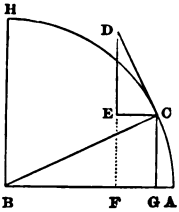

Fig. 12.

$$\text{ED}[= d(\sin x)] = \cos x dx,$$

and
$$-\text{EC}[= d(\cos x)] = -\sin x dx.$$

Hence, as ABC is any value of x, we have in general

$$d \sin x = \cos x dx,$$

and
$$d \cos x = -\sin x dx.$$

42. *The differential of the tangent of an angle is equal to the square of the secant of the angle into the differential of the angle.*

For
$$\tan x = \frac{\sin x}{\cos x};$$

$$\therefore d \tan x = \frac{\cos x d \sin x - \sin x d \cos x}{\cos^2 x}$$
$$= \frac{(\cos^2 x + \sin^2 x) dx}{\cos^2 x}$$
$$= \frac{dx}{\cos^2 x} = \sec^2 x dx.$$

43. *The differential of the cotangent of an angle is equal to minus the square of the cosecant of the angle into the differential of the angle.*

For
$$\cot x = \tan\left(\frac{\pi}{2} - x\right);$$

$$\therefore d \cot x = \sec^2\left(\frac{\pi}{2} - x\right) d\left(\frac{\pi}{2} - x\right) = -\text{cosec}^2 x dx. \qquad \S 42.$$

44. *The differential of the secant of an angle is equal to the secant of the angle into the tangent of the angle into the differential of the angle.*

For $$\sec x = \frac{1}{\cos x};$$

$$\therefore d \sec x = \frac{-d \cos x}{\cos^2 x} = \frac{\sin x\,dx}{\cos^2 x} = \sec x \tan x\,dx.$$

45. *The differential of the cosecant of an angle is equal to minus the cosecant of the angle into the cotangent of the angle into the differential of the angle.*

For $$\operatorname{cosec} x = \sec\left(\frac{\pi}{2} - x\right);$$

$$\therefore d \operatorname{cosec} x = \sec\left(\frac{\pi}{2} - x\right)\tan\left(\frac{\pi}{2} - x\right)d\left(\frac{\pi}{2} - x\right) \qquad \S\ 44.$$

$$= - \operatorname{cosec} x \cot x\,dx.$$

46. $d \operatorname{vers} x = d(1 - \cos x) = \sin x\,dx.$

47. $d \operatorname{covers} x = d(1 - \sin x) = -\cos x\,dx.$

48. To prove these theorems by the method of limits, we need the following lemma, which is very useful also in the theory of curves.

LEMMA. *The limit of the ratio of an arc of any plane curve to its chord is unity.*

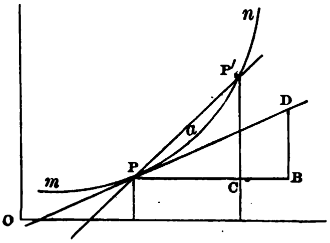

Fig. 13.

If s represents the length of the curve mn, and PB $= dx$, PD $= ds$; and if PC $= \Delta x$, arc PaP′ $= \Delta s$. Since s is a function of x, we have

$$\underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\Delta s}{\Delta x}\right] = \frac{ds}{dx}.$$

But $$\underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\text{chord PP}'}{\Delta x}\right] = \frac{ds}{dx},$$

since $$\underset{\Delta x \doteq 0}{\text{limit}}\left[\sec \text{CPP}'\right] = \sec \text{BPD}.$$

Hence, by division, we have $$\underset{\Delta s \doteq 0}{\text{limit}}\left[\frac{\Delta s}{\text{chord PP}'}\right] = 1. \qquad \S\ 9.$$

Cor. Since one-half of the chord of an arc whose radius is unity is the sine of half the arc,

$$\underset{x \doteq 0}{\text{limit}}\left[\frac{\sin x}{x}\right] = 1.$$

49. *To prove that* $d \sin x = \cos x\, dx$ *by the method of limits.*

Let $y = \sin x;$

then $\Delta y = \sin(x + \Delta x) - \sin x.$

But, from Trigonometry, we have

$$\sin x - \sin y = 2 \cos \tfrac{1}{2}(x + y) \sin \tfrac{1}{2}(x - y).$$

$$\therefore \Delta y = 2 \cos(x + \tfrac{1}{2}\Delta x) \sin \tfrac{1}{2}\Delta x.$$

$$\therefore \frac{\Delta y}{\Delta x} = \cos(x + \tfrac{1}{2}\Delta x)\frac{\sin \frac{1}{2}\Delta x}{\frac{1}{2}\Delta x}.$$

Now $\underset{\Delta x \doteq 0}{\text{limit}}[\cos(x + \tfrac{1}{2}\Delta x)] = \cos x;$

and $\underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\sin \frac{1}{2}\Delta x}{\frac{1}{2}\Delta x}\right] = 1.$ § 48, Cor.

$$\therefore \frac{dy}{dx} = \cos x.$$

The other theorems can be proved in like manner.

Examples.

Differentiate:

1. $\sin ax.$ *Ans.* $a \cos ax\, dx.$

2. $y = \cos \frac{x}{a}.$ $\frac{dy}{dx} = -\frac{1}{a}\sin \frac{x}{a}.$

3. $y = \cos x^3 = \cos(x^3).$ $dy = -3x^2 \sin x^3 dx.$

4. $f(x) = \tan^m x = (\tan x)^m.$ $f'(x)dx = m \tan^{m-1} x \sec^2 x\, dx.$

5. $f(x) = \tan x + \sec x.$ $f'(x) = \frac{1 + \sin x}{\cos^2 x}.$

6. $y = \sin(\log x).$ $\frac{dy}{dx} = \frac{\cos(\log x)}{x}.$

7. $y = \log(\tan x)$. $\quad \dfrac{dy}{dx} = \dfrac{2}{2 \sin x \cos x} = \dfrac{2}{\sin 2x}$.

8. $y = \log(\sin x)$. $\quad \dfrac{dy}{dx} = \cot x$.

9. $y = \log(\cot x)$. $\quad \dfrac{dy}{dx} = -\dfrac{2}{\sin 2x}$.

10. $y = \dfrac{1 - \tan x}{\sec x} [= \cos x - \sin x]$.

11. $y = x^n e^{\sin x}$. $\quad dy = x^{n-1} e^{\sin x}(n + x \cos x)dx$.

12. $y = \sin(nx)\sin^n x$. $\quad dy = n \sin^{n-1} x \sin(nx + x)dx$.

13. $y = e^x \log \sin x$. $\quad dy = e^x(\cot x + \log \sin x)dx$.

14. $y = \tan(\log x)$.

15. $y = \log \sec x$.

16. $y = \dfrac{\cos x}{2 \sin^2 x}$.

17. $y = 4 \sin^m ax$. $\quad dy = 4am \sin^{m-1} ax \cos ax\, dx$.

18. $y = x^{\sin x}$. $\quad \dfrac{dy}{dx} = x^{\sin x}\left(\dfrac{\sin x}{x} + \log x \cos x\right)$.

19. $y = (\sin x)^{\tan x}$. $\quad \dfrac{dy}{dx} = (\sin x)^{\tan x}(1 + \sec^2 x \log \sin x)$.

20. $y = \tan a^{\frac{1}{x}}$. $\quad \dfrac{dy}{dx} = -\dfrac{a^{\frac{1}{x}} \sec^2 a^{\frac{1}{x}} \log a}{x^2}$.

21. $y = \dfrac{\tan^3 x}{3} - \tan x + x$. $\quad \dfrac{dy}{dx} = \tan^4 x$.

22. $y = e^{(a+x)^2} \sin x$. $\quad \dfrac{dy}{dx} = e^{(a+x)^2}[2(a+x)\sin x + \cos x]$.

23. $y = e^{-a^2x^2} \cos rx$. $\quad \dfrac{dy}{dx} = -e^{-a^2x^2}(2a^2x \cos rx + r \sin rx)$.

24. $y = \tan\sqrt{1 - x}$. $\quad \dfrac{dy}{dx} = \dfrac{-(\sec\sqrt{1 - x})^2}{2\sqrt{1 - x}}$.

25. Are $\sin x$, $\cos x$, and $\tan x$ increasing or decreasing functions of x?

$d \cos x = -\sin x \, dx$; and $-\sin x$ is positive when x is of the third or the fourth quadrant, and negative when x is of the first or the second; hence, $\cos x$ is an increasing function when x is of the third or the fourth quadrant, and a decreasing function when x is of the first or the second. $d \tan x = \sec^2 x \, dx$, and $\sec^2 x$ is always positive; hence $\tan x$ is an increasing function of x between its limits of continuity; that is, between $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$, etc.

26. At what values of x does $\sin x$ change as fast as x? At what values does $\cos x$? $\tan x$? $\cot x$?

Ans. $\sin x$ does, when $x = 0$ and π;

$\cos x$ does, when $x = \frac{\pi}{2}$ and $\frac{3}{2}\pi$.

27. If the change of x and $\cos x$ became uniform at 30°, how much would $\cos x$ decrease while x increases from 30° to 30° 15′?

Let $y = \cos x$; then $dy = -\sin x dx = -\frac{1}{2} dx$, when $x = 30°$.

Let $dx = 15' = \frac{3.14159}{180 \times 4} = .004363$; then $dy = -.002182$.

Hence $\cos x$ would decrease .002182. This is evidently less than the actual decrement.

28. A vertical wheel whose circumference is 20 ft. makes 5 revolutions a second about a fixed axis. How fast is a point in its circumference moving horizontally, when it is 30° from either extremity of the horizontal diameter?

Ans. 50 ft. a second.

29. What is the slope of the curve $y = \sin x$? Its inclination lies between what values? What is its inclination at $x = 0$? What at $x = \frac{\pi}{2}$?

The slope $= \cos x$; hence, at any point, it must be something between -1 and $+1$ inclusive. Hence, the inclination of the curve at any point is something between 0 and $\frac{\pi}{4}$, or something between $\frac{3}{4}\pi$ and π inclusive.

Ans. $\frac{\pi}{4}$; 0.

30. What is the slope of the curve $y = \tan x$? Its inclination lies between what values? What is its inclination at $x = 0$? What at $x = \frac{\pi}{4}$?

Ans. $\sec^2 x$; between $\frac{\pi}{4}$ and $\frac{\pi}{2}$ inclusive; $\frac{\pi}{4}$; 63° 26′ 6″.

Anti-Trigonometric Functions.

50. $d(\sin^{-1}x)^* = \frac{dx}{\sqrt{1-x^2}}$.

Let $y = \sin^{-1}x$, then $x = \sin y$;

$$\therefore dx = \cos y\, dy = \sqrt{1-\sin^2 y}\, dy = \sqrt{1-x^2}\, dy.$$

$$\therefore dy\, [= d(\sin^{-1}x)] = \frac{dx}{\sqrt{1-x^2}}.$$

51. $d(\cos^{-1}x) = d\left(\frac{\pi}{2} - \sin^{-1}x\right) = -\frac{dx}{\sqrt{1-x^2}}$. § 50.

52. $d(\tan^{-1}x) = \frac{dx}{1+x^2}$.

Let $y = \tan^{-1}x$, then $x = \tan y$;

$$\therefore dx = \sec^2 y\, dy = (1 + \tan^2 y)dy = (1 + x^2)dy.$$

$$\therefore dy\, [= d(\tan^{-1}x)] = \frac{dx}{1+x^2}.$$

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[*] To avoid the ambiguity of the double sign ±, we shall, in these formulas, limit $\sin^{-1}x$, $\cos^{-1}x$, etc., to values between 0 and $\frac{\pi}{2}$. They may be made general, however, by writing the double sign in the second member of each.

53. $d(\cot^{-1}x) = d\left(\frac{\pi}{2} - \tan^{-1}x\right) = -\frac{dx}{1+x^2}.$

54. $d(\sec^{-1}x) = \frac{dx}{x\sqrt{x^2-1}}.$

Let $y = \sec^{-1}x$, then $x = \sec y$;

$$\therefore dx = \sec y \tan y\, dy = x\sqrt{x^2-1}\, dy.$$

$$\therefore dy\, [= d(\sec^{-1}x)] = \frac{dx}{x\sqrt{x^2-1}}.$$

55. $d(\operatorname{cosec}^{-1}x) = d\left(\frac{\pi}{2} - \sec^{-1}x\right) = -\frac{dx}{x\sqrt{x^2-1}}.$

56. $d(\operatorname{vers}^{-1}x) = \frac{dx}{\sqrt{2x-x^2}}.$

Let $y = \operatorname{vers}^{-1}x$, then $x = \operatorname{vers} y$;

$$\begin{aligned}\therefore dx = \sin y\, dy &= \sqrt{1-\cos^2 y}\, dy \\ &= \sqrt{1-(1-\operatorname{vers} y)^2}\, dy \\ &= \sqrt{2\operatorname{vers} y - \operatorname{vers}^2 y}\, dy \\ &= \sqrt{2x-x^2}\, dy.\end{aligned}$$

$$\therefore dy\, [= d(\operatorname{vers}^{-1}x)] = \frac{dx}{\sqrt{2x-x^2}}.$$

57. $d(\operatorname{covers}^{-1}x) = d\left(\frac{\pi}{2} - \operatorname{vers}^{-1}x\right) = -\frac{dx}{\sqrt{2x-x^2}}.$

EXAMPLES.

1. Prove that $d\left(\sin^{-1}\frac{x}{a}\right) = \frac{dx}{\sqrt{a^2-x^2}}.$

$$d\left(\sin^{-1}\frac{x}{a}\right) = \frac{d\left(\frac{x}{a}\right)}{\sqrt{1-\left(\frac{x}{a}\right)^2}} = \frac{dx}{\sqrt{a^2-x^2}}. \qquad \S\ 50.$$

2. Prove that $d\left(\cos^{-1}\frac{x}{a}\right)=-\frac{dx}{\sqrt{a^2-x^2}}$; $d\left(\tan^{-1}\frac{x}{a}\right)=\frac{adx}{a^2+x^2}$;

$$d\left(\cot^{-1}\frac{x}{a}\right)=-\frac{adx}{a^2+x^2}; \qquad d\left(\sec^{-1}\frac{x}{a}\right)=\frac{adx}{x\sqrt{x^2-a^2}};$$

$$d\left(\operatorname{cosec}^{-1}\frac{x}{a}\right)=-\frac{adx}{x\sqrt{x^2-a^2}}; \qquad d\left(\operatorname{vers}^{-1}\frac{x}{a}\right)=\frac{dx}{\sqrt{2ax-x^2}};$$

$$d\left(\operatorname{covers}^{-1}\frac{x}{a}\right)=-\frac{dx}{\sqrt{2ax-x^2}}.$$

3. $y=x\sin^{-1}x.$ $\qquad \frac{dy}{dx}=\sin^{-1}x+\frac{x}{\sqrt{1-x^2}}.$

4. $y=\tan x\tan^{-1}x.$ $\qquad \frac{dy}{dx}=\sec^2x\tan^{-1}x+\frac{\tan x}{1+x^2}.$

5. $y=\tan^{-1}\frac{2x}{1+x^2}.$ $\qquad \frac{dy}{dx}=\frac{2(1-x^2)}{1+6x^2+x^4}.$

6. $y=\sin^{-1}\frac{x+1}{\sqrt{2}}.$ $\qquad \frac{dy}{dx}=\frac{1}{\sqrt{1-2x-x^2}}.$

7. $y=\sec^{-1}\frac{1}{2x^2-1}.$ $\qquad \frac{dy}{dx}=-\frac{2}{\sqrt{1-x^2}}.$

8. $y=\cos^{-1}\frac{x^{2n}-1}{x^{2n}+1}.$ $\qquad \frac{dy}{dx}=-\frac{2nx^{n-1}}{x^{2n}+1}.$

9. $y=\tan^{-1}(n\tan x).$ $\qquad \frac{dy}{dx}=\frac{n}{\cos^2x+n^2\sin^2x}.$

10. $y=x^{\sin^{-1}x}.$ $\qquad \frac{dy}{dx}=x^{\sin^{-1}x}\left(\frac{\sin^{-1}x}{x}+\frac{\log x}{\sqrt{1-x^2}}\right).$

11. A wheel whose radius is r rolls along a horizontal line with a velocity v'; required the velocity of any point P in its circumference; also the velocity of P horizontally and vertically.

Let APX (Fig. 14) represent the cycloid traced by the point P, referred to the axes AX and AY; then will the horizontal and vertical velocities of P be the rates of change of x and y respectively. D being the point of contact, AD $= r \operatorname{vers}^{-1}\frac{y}{r}$. Since the centre is vertically over D, its velocity is equal to the rate of increase of AD;

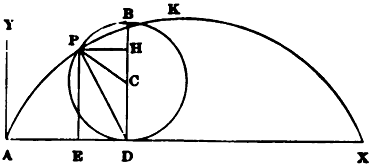

Fig. 14.

$$\therefore v' = \frac{d^{*}}{dt}\left(r \operatorname{vers}^{-1}\frac{y}{r}\right) = \frac{r}{\sqrt{2ry-y^2}}\frac{dy}{dt}.$$

$$\therefore \frac{dy}{dt} = \frac{\sqrt{2ry-y^2}}{r}v' = \text{the velocity vertically.} \qquad (1)$$

Since AE = AD − PH, and PH $= \sqrt{y(2r-y)}$, the equation of the cycloid is evidently

$$x = r\operatorname{vers}^{-1}\frac{y}{r} - \sqrt{2ry-y^2}.$$

$$\therefore dx = \frac{y}{\sqrt{2ry-y^2}}dy.$$

Dividing by dt, and substituting the value of $\frac{dy}{dt}$, we have

$$\frac{dx}{dt} = \frac{y}{\sqrt{2ry-y^2}}\frac{dy}{dt} = \frac{y}{r}v' = \text{the velocity horizontally.} \qquad (2)$$

From § 16, Cor. 2, and equations (1) and (2), we have

$$ds = \sqrt{dx^2+dy^2} = \left(\frac{2ry-y^2}{r^2}+\frac{y^2}{r^2}\right)^{\frac{1}{2}} v'dt = \sqrt{\frac{2y}{r}}\,v'dt.$$

$$\therefore \frac{ds}{dt} = \sqrt{\frac{2y}{r}}\,v' = \text{the velocity of P;} \qquad (3)$$

for the velocity of P equals the rate of increase of s.

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* The symbol $\frac{d}{dt}$ indicates the operation of taking the derivative with respect to t.

From (1), (2), and (3), we have,

if $y = 0,\ \frac{dy}{dt} = 0,\ \frac{dx}{dt} = 0,$ and $\frac{ds}{dt} = 0;$

if $y = r,\ \frac{dy}{dt} = v',\ \frac{dx}{dt} = v',$ and $\frac{ds}{dt} = \sqrt{2}\,v';$

if $y = 2r,\ \frac{dy}{dt} = 0,\ \frac{dx}{dt} = 2v',$ and $\frac{ds}{dt} = 2v'.$

Hence, when a point of the circumference is in contact with the line, its velocity is zero; when it is in the same horizontal plane as the centre, its velocity horizontally and vertically is the same as the velocity of the centre; and when it is at the highest point, its motion is entirely horizontal, and its velocity is twice that of the centre.

Since $$\frac{ds}{dt} = \sqrt{\frac{2y}{r}}\,v' = \frac{\sqrt{2ry}}{r}\,v';$$

$$\therefore \frac{ds}{dt} : v' :: \sqrt{2ry} : r.$$

Hence the velocity of P is to that of C as the chord DP is to the radius DC; that is, P and C are momentarily moving about D with equal angular velocities.

MISCELLANEOUS EXAMPLES.

1. If $y = f(x)$, show that

$$\underset{\Delta x \doteq 0}{\text{limit}} \left[\frac{\Delta y}{\Delta x}\right] = \underset{\Delta x \doteq 0}{\text{limit}} \left[\frac{f(x + \Delta x) - f(x)}{\Delta x}\right] = f'(x).$$

2. Find $\frac{d}{dx}\left(\frac{x}{\sqrt{a^2 - x^2}}\right)$. *Ans.* $\frac{a^2}{(a^2 - x^2)^{\frac{3}{2}}}$.

3. Find $\frac{d}{dx}\left[\frac{x^3}{(1 - x^2)^{\frac{3}{2}}}\right]$. $\frac{3x^2}{(1 - x^2)^{\frac{5}{2}}}$.

4. $f(x) = (x-3)e^{2x} + 4xe^{x} + x + 3.$
$$f'(x) = (2x-5)e^{2x} + 4(x+1)e^{x} + 1.$$

5. $y = \log \tan^{-1} x.$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{(1+x^2)\tan^{-1}x}.$

6. Find $\dfrac{d}{dx}[x^3 - 3\log(1+x^3)^{\frac{1}{3}}].$ $\qquad \dfrac{3x^5}{1+x^3}.$

7. $y = \dfrac{x\log x}{1-x} + \log(1-x).$ $\qquad \dfrac{dy}{dx} = \dfrac{\log x}{(1-x)^2}.$

8. $y = \dfrac{\sqrt{x^2+a^2} + \sqrt{x^2+b^2}}{\sqrt{x^2+a^2} - \sqrt{x^2+b^2}}.$
$$\frac{dy}{dx} = \frac{2x}{a^2-b^2}\left(2 + \sqrt{\frac{x^2+a^2}{x^2+b^2}} + \sqrt{\frac{x^2+b^2}{x^2+a^2}}\right).$$

Rationalize the denominator before differentiating.

9. $y = \dfrac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}-x}.$ $\qquad \dfrac{dy}{dx} = 2\left(2x + \dfrac{2x^2+1}{\sqrt{x^2+1}}\right).$

10. $y = \sqrt{\dfrac{1-x^2}{(1+x^2)^3}}.$ $\qquad \dfrac{dy}{dx} = \dfrac{-2x(2-x^2)}{\sqrt{1-x^2}\sqrt{(1+x^2)^5}}.$

11. $y = (x+\sqrt{1-x^2})^n.$ $\qquad \dfrac{dy}{dx} = n(x+\sqrt{1-x^2})^{n-1}\dfrac{\sqrt{1-x^2}-x}{\sqrt{1-x^2}}.$

12. $y = \log(\sqrt{1+x^2}+\sqrt{1-x^2}).$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{x}\left(1 - \dfrac{1}{\sqrt{1-x^4}}\right).$

13. $y = \dfrac{(\sin nx)^m}{(\cos mx)^n}.$ $\qquad \dfrac{dy}{dx} = \dfrac{mn(\sin nx)^{m-1}\cos(mx-nx)}{(\cos mx)^{n+1}}.$

14. $y = \dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}.$ $\qquad \dfrac{dy}{dx} = -\dfrac{2}{x^3}\left(1 + \dfrac{1}{\sqrt{1-x^4}}\right).$

15. $y = \tan^{-1}\dfrac{x}{a} + \log\sqrt{\dfrac{x-a}{x+a}}.$ $\qquad \dfrac{dy}{dx} = \dfrac{2ax^2}{x^4-a^4}.$

16. $y = \sec^{-1}\dfrac{a}{\sqrt{a^2 - x^2}}.$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{a^2 - x^2}}.$

17. $f(x) = (a^2 + x^2)\tan^{-1}\dfrac{x}{a}.$ $\qquad f'(x) = 2x\tan^{-1}\dfrac{x}{a} + a.$

18. $y = \sqrt{1 - x^2}\sin^{-1}x - x.$ $\qquad \dfrac{dy}{dx} = -\dfrac{x\sin^{-1}x}{\sqrt{1 - x^2}}.$

19. $y = \sin^{-1}\dfrac{1 - x^2}{1 + x^2}.$ $\qquad \dfrac{dy}{dx} = -\dfrac{2}{1 + x^2}.$

20. Find $\dfrac{d}{dx}(e^{ax}\sin^m rx).$

$$f'(x) = e^{ax}\sin^{m-1}rx(a\sin rx + mr\cos rx).$$

21. $y = \log(2x - 1 + 2\sqrt{x^2 - x - 1}).$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{x^2 - x - 1}}.$

22. $y = \cos^{-1}\dfrac{3 + 5\cos x}{5 + 3\cos x}.$ $\qquad dy = \dfrac{4\,dx}{5 + 3\cos x}.$

23. $f(x) = e^{(a+x)^2}\sin x.$ $\quad f'(x) = e^{(a+x)^2}[2(a + x)\sin x + \cos x].$

24. $y = \log[\log(a + bx^n)].$ $\qquad dy = \dfrac{nbx^{n-1}dx}{(a + bx^n)\log(a + bx^n)}.$

25. $y = \log\left(\dfrac{1 + x}{1 - x}\right)^{\frac{1}{4}} - \frac{1}{2}\tan^{-1}x.$ $\qquad dy = \dfrac{x^2dx}{1 - x^4}.$

26. $y = \sin^{-1}\dfrac{3 + 2x}{\sqrt{13}}.$ $\qquad dy = \dfrac{dx}{\sqrt{1 - 3x - x^2}}.$

27. $y = e^{x^x}\tan^{-1}x.$ $\quad \dfrac{dy}{dx} = e^{x^x}\left[\dfrac{1}{1 + x^2} + x^x\tan^{-1}x(1 + \log x)\right].$

28. $x = e^{\frac{x-y}{y}}.$

$$\frac{x - y}{y} = \log x; \quad \therefore y = \frac{x}{1 + \log x}; \quad \therefore \frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}.$$

29. $$y = \cfrac{x^2}{1 + \cfrac{x^2}{1 + \cfrac{x^2}{1 + \text{etc. to infinity.}}}}$$

$$y = \frac{x^2}{1+y}; \quad \therefore dy = \pm \frac{x\,dx}{\sqrt{x^2+\frac{1}{4}}}.$$

30. $y = \log(x + \sqrt{x^2 - a^2}) + \sec^{-1}\frac{x}{a}$. $\qquad \frac{dy}{dx} = \frac{1}{x}\sqrt{\frac{x+a}{x-a}}$.

31. $y = \log\frac{\sqrt{1-x^2}+x\sqrt{2}}{\sqrt{1-x^2}}$. $\qquad \frac{dy}{dx} = \frac{\sqrt{2}}{(\sqrt{1-x^2}+x\sqrt{2})(1-x^2)}$.

32. $y = \log\sqrt{\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}}$. $\qquad \frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$.

$$y = \tfrac{1}{2}\log\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x} = \log(\sqrt{1+x^2}+x).$$

33. Given $f(x) = 3x^2 - x + 6$; to find $f(y)$, $f(a)$, $f(2)$, $f(0)$, $f(x+y)$.

When, in connection with the symbol $f(x)$, expressions like $f(y)$, $f(a)$, $f(2)$, $f(0)$, $f(x+y)$ are employed, they denote respectively the results obtained by substituting y, a, 2, 0, and $x+y$ for x in the value of $f(x)$. Thus here $f(y) = 3y^2 - y + 6$, $f(a) = 3a^2 - a + 6$, $f(2) = 16$, etc.

34. Given $\phi(x) = 4x^3 - x^2$; find $\phi(z)$, $\phi(b)$, $\phi(1)$, $\phi(3)$, $\phi(0)$. Given $f(x+y) = a^{x+y}$; to find $f(x)$, $f(y)$, $f(5)$.

35. Find $df(a+x)$. $\qquad f'(a+x)d(a+x)$, or $f'(a+x)dx$.

36. Find $df(ax)$. $\qquad f'(ax)a\,dx$.

37. Find $\frac{d}{dx}f\left(\frac{x^2}{a}\right)$. $\qquad f'\left(\frac{x^2}{a}\right)\frac{2x}{a}$.

38. Find $df(x+y)$. $\qquad f'(x+y)(dx+dy)$.

CHAPTER III.

INTEGRATION.

58. A function or variable is called the **Integral** of its differential. Thus, x^3 is the integral of $3x^2dx$; and $f(x)+C$, C being any constant, is the integral of $f'(x)dx$.

Integration is the operation of finding the integral of a differential.

The problem of differentiation and the *inverse* problem of integration may be stated also as follows:

That of Differentiation, or of the Differential Calculus, is, *To find the ratio of the rates of change of a function and its variable.*

That of Integration, or of the Integral Calculus, is, *Having given the ratio of the rates of change of a function and its variable, to find the function.*

The sign of integration is $\int$. Thus, $\int$ in $\int 4x^3dx$ indicates the operation of integrating $4x^3dx$. Hence, d and $\int$, as signs of operation, neutralize each other. For example, $\int d(x^3)=x^3$, and $d\int 3x^2dx=3x^2dx$. The whole expression $\int 4x^3dx$, read "the integral of $4x^3dx$," represents the integral of $4x^3dx$.

59. Elementary Principles.

I. Since $dC=0$, C being any constant, $\int 0=C$.

Hence, as 0 may be added to any differential, the *general* form of its integral will contain an *indeterminate* constant term.

In the Applications of the Calculus, this constant term is eliminated, or determined from the data of the problem.

II. Since $d(ay+ac)=a\,dy$,

$$\therefore \int a\,dy=\int d(ay+ac)=a(y+c)$$
$$=a\int d(y+c)=a\int dy.$$

Hence, *a constant factor can be moved from one side of the sign of integration to the other without changing the value of the integral.*

III. Since $d(x - y + z + c) = dx - dy + dz$,

$$\begin{aligned} \therefore \int (dx - dy + dz) &= x - y + z + c \\ &= \int dx - \int dy + \int d(z + c) \\ &= \int dx - \int dy + \int dz. \end{aligned}$$

Hence, *the integral of a sum of terms is equal to the sum of the integrals of the terms.*

60. Fundamental Formulas. Since integration is the *inverse* of differentiation, general formulas for integration may be obtained by *reversing* the general formulas for differentiation.

1. $\int \frac{dx}{x} = \log x + \log c,$[*] $\qquad \because d(\log x + \log c) = \frac{dx}{x}.$

2. $\int ax^n dx = \frac{ax^{n+1}}{n+1} + C, \qquad \because d\left(\frac{ax^{n+1}}{n+1} + C\right) = ax^n dx.$

3. $\int a^x \log a\, dx = a^x + C, \qquad \because d(a^x + C) = a^x \log a\, dx.$

   $\int e^x dx = e^x + C.$

4. $\int \cos x\, dx = \sin x + C, \qquad \because d(\sin x + C) = \cos x\, dx.$

5. $\int -\sin x\, dx = \cos x + C.$

6. $\int \sec^2 x\, dx = \tan x + C.$

7. $\int -\operatorname{cosec}^2 x\, dx = \cot x + C.$

8. $\int \sec x \tan x\, dx = \sec x + C.$

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[*] When the integral is a logarithm, it is customary to write the indeterminate constant term as a logarithm.

9. $\int -\operatorname{cosec} x \cot x \, dx = \operatorname{cosec} x + C.$

*10. $\int \sin x \, dx = \operatorname{vers} x + C$, or $-\cos x + C'$.

*11. $\int -\cos x \, dx = \operatorname{covers} x + C$, or $-\sin x + C'$.

12. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C.$

*13. $\int \frac{-dx}{\sqrt{1-x^2}} = \cos^{-1} x + C$, or $-\sin^{-1} x + C'$.

14. $\int \frac{dx}{1+x^2} = \tan^{-1} x + C.$

*15. $\int \frac{-dx}{1+x^2} = \cot^{-1} x + C$, or $-\tan^{-1} x + C'$.

16. $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C.$

*17. $\int \frac{-dx}{x\sqrt{x^2-1}} = \operatorname{cosec}^{-1} x + C$, or $-\sec^{-1} x + C'$.

18. $\int \frac{dx}{\sqrt{2x-x^2}} = \operatorname{vers}^{-1} x + C.$

*19. $\int \frac{-dx}{\sqrt{2x-x^2}} = \operatorname{covers}^{-1} x + C$, or $-\operatorname{vers}^{-1} x + C'$.

The differentials in these nineteen formulas are the *fundamental integrable forms*, to one of which we endeavor to reduce

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* Two integrals having the same or equal differentials must change at the same rate; hence they must be equal, or have a constant difference. The constant difference between the variable terms of the integrals in the last four starred formulas is evidently $\frac{\pi}{2}$; for, when $x < \frac{\pi}{2}$,

$$\cos^{-1} x + \sin^{-1} x = \frac{\pi}{2}, \quad \cot^{-1} x + \tan^{-1} x = \frac{\pi}{2}, \text{ etc.}$$

The starred formulas are not necessary, since the second integral in each is given by a previous formula.

every differential that is to be integrated. The processes of the Integral Calculus are largely a succession of transformations and devices to effect this reduction.

61. To facilitate the application of formulas 1 and 2, they may be stated as follows:

I. *The integral of a fraction whose numerator is the differential of its denominator is the Naperian logarithm of the denominator, plus a constant.*

II. *Whenever a differential can be resolved into three factors, —viz., a constant factor, a factor which is a variable with any constant exponent except* −1, *and a factor which is the differential of the variable without its exponent,—its integral is the product of the constant factor into the variable with its exponent increased by* 1, *divided by the new exponent, plus a constant.*

EXAMPLES.

Find

1. $\int ax^6 dx.$	*Ans.* $\frac{ax^7}{7} + C.$
2. $\int bx^{\frac{1}{2}} dx.$	$\frac{2}{3} bx^{\frac{3}{2}} + C.$
3. $\int 2x^{\frac{2}{3}} dx.$	$\frac{6}{5} x^{\frac{5}{3}} + C.$
4. $\int (\frac{7}{2} ax^{\frac{5}{2}} - \frac{5}{2} bx^{\frac{3}{2}}) dx.$	$ax^{\frac{7}{2}} - bx^{\frac{5}{2}} + C.$
5. $\int \frac{4\,dx}{x^5} = \int 4x^{-5} dx.$	$-x^{-4} + C.$
6. $\int \left(\frac{12}{x^3} - \frac{5}{x^4}\right) dx.$	$-\frac{6}{x^2} + \frac{5}{3x^3} + C.$
7. $\int \frac{dx}{\sqrt{x}}.$	$2\sqrt{x} + C.$
8. $\int \left(bx^3 + \frac{1}{x^{\frac{3}{2}}}\right) dx.$	$\frac{bx^4}{4} - \frac{2}{\sqrt{x}} + C.$

9. $\int \frac{a\,dx}{x^n}$. $\qquad \frac{a}{(1-n)x^{n-1}} + C.$

10. $\int b(6ax^2 + 8bx^3)^{\frac{5}{3}}(2ax + 4bx^2)\,dx$.

Since $d(6ax^2 + 8bx^3) = (12ax + 24bx^2)\,dx$, we see that the differential factor $(2ax + 4bx^2)\,dx$ must be multiplied by 6 to make it the differential of the variable $6ax^2 + 8bx^3$.

$$\therefore \int b(6ax^2 + 8bx^3)^{\frac{5}{3}}(2ax + 4bx^2)\,dx$$

$$= \int \frac{b}{6}(6ax^2 + 8bx^3)^{\frac{5}{3}}(12ax + 24bx^2)\,dx$$

$$= \frac{b}{16}(6ax^2 + 8bx^3)^{\frac{8}{3}} + C. \qquad \text{§ 61, II.}$$

11. $\int [a(ax + bx^2)^{\frac{1}{3}}dx + 2b(ax + bx^2)^{\frac{1}{3}}x\,dx]$.

$$\int [a(ax + bx^2)^{\frac{1}{3}}dx + 2b(ax + bx^2)^{\frac{1}{3}}x\,dx]$$

$$= \int (ax + bx^2)^{\frac{1}{3}}(a + 2bx)\,dx = \tfrac{3}{4}(ax + bx^2)^{\frac{4}{3}} + C.$$

12. $\int (2a + 3bx)^3\,dx$. $\qquad \frac{1}{12b}(2a + 3bx)^4 + C.$

13. $\int \frac{x^2\,dx}{(a^2 + x^3)^{\frac{1}{2}}}$. $\qquad \tfrac{2}{3}(a^2 + x^3)^{\frac{1}{2}} + C.$

14. $\int (1 + \tfrac{9}{4}x)^{\frac{1}{2}}\,dx$. $\qquad \tfrac{8}{27}(1 + \tfrac{9}{4}x)^{\frac{3}{2}} + C.$

15. $\int (6x^4 + 2x^2 - 5)(3x^2 - 1)\,dx$. $\qquad \frac{18x^7}{7} - \frac{17x^3}{3} + 5x + C.$

16. $\int \frac{dx}{x - a}$. $\qquad \log(x - a) + \log c$, or $\log[(x - a)c]$.

17. $\int \frac{x^{n-1}\,dx}{a + bx^n}$.

$$\int \frac{x^{n-1}\,dx}{a + bx^n} = \frac{1}{nb}\int \frac{nb\,x^{n-1}\,dx}{a + bx^n} = \frac{1}{nb}\log(a + bx^n) + \log c.$$

18. $\int \frac{5x^2\,dx}{10x^3 + 15}$. $\qquad \log[(10x^3 + 15)^{\frac{1}{6}}c].$

19. $\int \frac{2\,bx^2\,dx}{ae+bx^3}$. $\log\left[(ae+bx^3)^{\frac{2}{3}}c\right]$.

20. $\int \frac{5\,bx\,dx}{8\,a-6\,bx^2}$. $\log \frac{c}{(8\,a-6\,bx^2)^{\frac{5}{12}}}$.

21. $\int \frac{5(2\,a-x^2)^3\,dx}{x^5}$. $5\left[-\frac{2\,a^3}{x^4}+\frac{6\,a^2}{x^2}+6\,a\log x-\frac{1}{2}x^2\right]+C$.

22. $\int (b-x^2)^3 x^{\frac{1}{2}}\,dx$. $\frac{2}{3}b^3x^{\frac{3}{2}}-\frac{6}{7}b^2x^{\frac{7}{2}}+\frac{6}{11}bx^{\frac{11}{2}}-\frac{2}{15}x^{\frac{15}{2}}+C$.

23. $\int \cot x\,dx\left[=\int\frac{\cos x\,dx}{\sin x}\right]$. $\log(c\sin x)$.

24. $\int \frac{dx}{x\log x}\left[=\int\frac{\frac{dx}{x}}{\log x}\right]$. $\log(\log x)+\log c$.

25. $\int \sqrt{2\,px}\,dx$. $\frac{2}{3}x\sqrt{2\,px}+C$.

26. $\int 2\,\pi y\left(\frac{y^2}{p^2}+1\right)^{\frac{1}{2}}dy$. $\frac{2\,\pi}{3\,p}(y^2+p^2)^{\frac{3}{2}}+C$.

27. $\int \frac{-(2\,ax-x^2)\,dx}{(3\,ax^2-x^3)^{\frac{1}{3}}}$. $-\frac{1}{2}(3\,ax^2-x^3)^{\frac{2}{3}}+C$.

28. $\int (2\,x^4-3\,x^2+1)^{\frac{1}{2}}(x^3-\frac{3}{4}x)\,dx$. $\frac{1}{12}(2\,x^4-3\,x^2+1)^{\frac{3}{2}}+C$.

29. $\int \sin x\cos x\,dx$. $\frac{1}{2}\sin^2 x+C$.

30. $\int (\log x)^3\frac{dx}{x}$. $\frac{1}{4}(\log x)^4+\log c$.

31. $\int a^{3x}\log a\,dx\left[=\frac{1}{3}\int a^{3x}\log a\,3\,dx\right]$. $\frac{1}{3}a^{3x}+C$.

32. $\int (\log x)^m\frac{dx}{x}$. $\frac{1}{m+1}(\log x)^{m+1}+\log c$.

33. $\int a^{4x}\,dx$. $\frac{1}{4\log a}a^{4x}+C$.

34. $\int e^{\frac{x}{n}}dx.$ $\qquad ne^{\frac{x}{n}}+C.$

35. $\int \frac{\sin\theta\, d\theta}{\cos^2\theta}\left[=\int \tan\theta\sec\theta\, d\theta\right].$ $\qquad \sec\theta + C.$

36. $\int ae^{bx}dx.$ $\qquad \frac{a}{b}e^{bx}+C.$

37. $\int ca^{2x}dx.$ $\qquad \frac{c}{2\log a}a^{2x}+C.$

38. $\int \cos(mx)\,dx\left[=\frac{1}{m}\int \cos(mx)\,m\,dx\right].$ $\qquad \frac{1}{m}\sin(mx)+C.$

39. $\int \sec^2(mx)\,dx.$ $\qquad \frac{1}{m}\tan(mx)+C.$

40. $\int \sin^4 x\cos x\,dx.$ $\qquad \frac{1}{5}\sin^5 x + C.$

41. $\int \sin^3(2x)\cos(2x)\,dx.$ $\qquad \frac{1}{8}\sin^4(2x)+C.$

42. $\int \cos^4(3x)\sin(3x)\,dx.$ $\qquad -\frac{1}{15}\cos^5(3x)+C.$

43. $\int \sec^2(x^3)x^2\,dx.$ $\qquad \frac{1}{3}\tan x^3 + C.$

44. $\int 7\sec^2(x^2)x\,dx.$ $\qquad \frac{7}{2}\tan x^2 + C.$

45. $\int \log x\frac{dx}{x}.$ $\qquad \frac{1}{2}(\log x)^2+\log c.$

46. $\int \frac{\sin x\,dx}{a+b\cos x}\left[=-\frac{1}{b}\int\frac{-b\sin x\,dx}{a+b\cos x}\right].$ $\qquad \log\frac{c}{(a+b\cos x)^{\frac{1}{b}}}.$

47. $\int 5\sec(3x)\tan(3x)\,dx.$ $\qquad \frac{5}{3}\sec(3x)+C.$

48. $\int 5\cos(a+bx)\,dx.$ $\qquad \frac{5}{b}\sin(a+bx)+C.$

49. $\int 4\,\text{cosec}(ax)\cot(ax)\,dx.$ $\qquad -\frac{4}{a}\text{cosec}(ax)+C.$

50. $\int e^{\cos x}\sin x\,dx.$ $\qquad -e^{\cos x}+C.$

51. $\int e^{2\sin x}\cos x\,dx.$ $\qquad \frac{1}{2}e^{2\sin x}+C.$

52. $\int\frac{(1+\cos x)\,dx}{x+\sin x}.$ $\qquad \log\left[(x+\sin x)c\right].$

53. $\int\frac{x\,dx}{1+x^4}\left[=\frac{1}{2}\int\frac{2x\,dx}{1+(x^2)^2}\right].$ $\qquad \frac{1}{2}\tan^{-1}x^2+C.$

54. $\int(1+x)(1-x^2)x\,dx.$ $\qquad \frac{x^2}{2}-\frac{x^4}{4}+\frac{x^3}{3}-\frac{x^5}{5}+C.$

55. $\int\frac{x^2+1}{x-1}dx\left[=\int\left(x+1+\frac{2}{x-1}\right)dx\right].$

$\qquad \frac{x^2}{2}+x+2\log(x-1)+C.$

56. $\int\frac{x^4\,dx}{x^2+1}.$ $\qquad \frac{x^3}{3}-x+\tan^{-1}x+C.$

57. $\int\frac{x^{n-1}\,dx}{(a+bx^n)^m}.$ $\qquad \frac{(a+bx^n)^{1-m}}{bn(1-m)}+C.$

58. $\int\frac{5x^3\,dx}{3x^4+7}.$ $\qquad \log\left[c(3x^4+7)^{\frac{5}{12}}\right].$

62. Auxiliary Formulas. By integrating the equations in Examples 1 and 2 of § 57, and those in Examples 27 and 28 of § 39, we obtain the following *auxiliary* formulas for integration:

(a) $\int\frac{dx}{x^2+a^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}+C.$

(b) $\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\log\frac{x-a}{x+a}+\log c.$

(c) $\int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\frac{x}{a}+C.$

(d) $\int\frac{dx}{\sqrt{x^2\pm a^2}}=\log\left(x+\sqrt{x^2\pm a^2}\right)+\log c.$

(e) $$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + C.$$

(f) $$\int \frac{dx}{\sqrt{2ax-x^2}} = \text{vers}^{-1}\frac{x}{a} + C.$$

If the differentials in these formulas were negative, we would evidently have $\cos^{-1}\frac{x}{a}$, or $-\sin^{-1}\frac{x}{a}$, in place of $\sin^{-1}\frac{x}{a}$;

$$\log\frac{x+a}{x-a}, \text{ or } \log\frac{a+x}{a-x}, \text{ in place of } \log\frac{x-a}{x+a}; \text{ etc.}$$

EXAMPLES.

1. Deduce formulas (a), (c), (e), and (f) of § 62, from formulas 14, 12, 16, and 18 of § 60.

(c) $$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{\frac{dx}{a}}{\sqrt{1-\frac{x^2}{a^2}}} = \sin^{-1}\frac{x}{a} + C.$$

(e) $$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\int \frac{\frac{dx}{a}}{\frac{x}{a}\sqrt{\frac{x^2}{a^2}-1}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + C.$$

2. Deduce formulas (b) and (d) of § 62 from formula 1 of § 60.

Since $\frac{1}{x^2-a^2} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)$;

$$\therefore \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\int \frac{dx}{x-a} - \frac{1}{2a}\int \frac{dx}{x+a} = \frac{1}{2a}\log\frac{x-a}{x+a} + C.$$

To deduce (d), assume $\sqrt{x^2 \pm a^2} = z - x$.

$$\therefore \pm a^2 = z^2 - 2xz.$$

$$\therefore (z-x)\,dz = z\,dx, \text{ or } \frac{dz}{z} = \frac{dx}{z-x} = \frac{dx}{\sqrt{x^2 \pm a^2}}.$$

$$\therefore \int\frac{dx}{\sqrt{x^2\pm a^2}}=\int\frac{dz}{z}=\log(zc)=\log\left[\left(x+\sqrt{x^2\pm a^2}\right)c\right].$$

3. Find $\int\frac{dx}{c^2+b^2x^2}\left[=\frac{1}{b}\int\frac{d(bx)}{c^2+(bx)^2}\right]$. $\qquad \frac{1}{bc}\tan^{-1}\frac{bx}{c}+C.$

Here x of formula (**a**) $=bx$, and $a=c$.

4. $\int\frac{dx}{\sqrt{a^2c^2-b^2x^2}}\left[=\frac{1}{b}\int\frac{d(bx)}{\sqrt{(ac)^2-(bx)^2}}\right]$. $\qquad \frac{1}{b}\sin^{-1}\frac{bx}{ac}+C.$

5. $\int\frac{dx}{x\sqrt{c^2x^2-a^2b^2}}\left[=\int\frac{d(cx)}{(cx)\sqrt{(cx)^2-(ab)^2}}\right]$. $\qquad \frac{1}{ab}\sec^{-1}\frac{cx}{ab}+C.$

6. $\int\frac{-dx}{\sqrt{8cx-c^2x^2}}\left[=\frac{1}{c}\int\frac{-d(cx)}{\sqrt{8(cx)-(cx)^2}}\right]$. $\qquad \frac{1}{c}\text{covers}^{-1}\frac{cx}{4}+C.$

7. $\int\frac{dx}{\sqrt{7x^4-5x^2}}\left[=\int\frac{d(\sqrt{7}\,x)}{\sqrt{7}\,x\sqrt{7x^2-5}}\right]$. $\qquad \frac{1}{\sqrt{5}}\sec^{-1}\left(x\sqrt{\frac{7}{5}}\right)+C.$

8. $\int\frac{dx}{\sqrt{(1-x-x^2)}}\left[=\int\frac{dx}{\sqrt{\frac{5}{4}-\left(x+\frac{1}{2}\right)^2}}\right]$. $\qquad \sin^{-1}\frac{2x+1}{\sqrt{5}}+C.$

9. $\int\frac{dx}{\sqrt{(ax-x^2)}}\left[=\int\frac{dx}{\sqrt{\frac{a^2}{4}-\left(x-\frac{a}{2}\right)^2}}\right]$. $\qquad \sin^{-1}\frac{2x-a}{a}+C.$

10. $\int\frac{dx}{x^2+4x+5}\left[=\int\frac{dx}{(x+2)^2+1}\right]$. $\qquad \tan^{-1}(x+2)+C.$

11. $\int\frac{dx}{2x^2-2x+1}$. $\qquad \tan^{-1}(2x-1)+C.$

12. $\int\frac{dx}{x\sqrt{b^2x^2-a^2}}$. $\qquad \frac{1}{a}\sec^{-1}\frac{bx}{a}+C.$

13. $\int\frac{dx}{\sqrt{(2abx-b^2x^2)}}$. $\qquad \frac{1}{b}\text{vers}^{-1}\frac{bx}{a}+C.$

14. $\int\frac{x^{\frac{1}{2}}dx}{\sqrt{8-4x^3}}\left[=\frac{1}{3}\int\frac{\frac{3}{2}x^{\frac{1}{2}}dx}{\sqrt{2-(x^{\frac{3}{2}})^2}}\right].$ $\qquad\frac{1}{3}\sin^{-1}\sqrt{\frac{x^3}{2}}+C.$

15. $\int\frac{dx}{\sqrt{3x-x^2-2}}\left[=\int\frac{dx}{\sqrt{\frac{1}{4}-\left(x-\frac{3}{2}\right)^2}}\right].$ $\qquad\sin^{-1}(2x-3)+C.$

16. $\int\frac{dx}{\sqrt{1+x^2+x}}\left[=\int\frac{dx}{\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}}\right].$

$$\log\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)+\log c.$$

17. $\int\frac{x^2dx}{x^6-1}.$ $\qquad\frac{1}{6}\log\frac{x^3-1}{x^3+1}+\log c.$

18. $\int\frac{x\,dx}{a^4+x^4}.$ $\qquad\frac{1}{2a^2}\tan^{-1}\frac{x^2}{a^2}+C.$

19. $\int\frac{dx}{1+x+x^2}.$ $\qquad\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}+C.$

20. $\int\frac{-dx}{\sqrt{n(c^2-x^2)}}.$ $\qquad n^{-\frac{1}{2}}\cos^{-1}\frac{x}{c}+C.$

21. $\int\frac{dx}{\sqrt{ax-x^2}}.$ $\qquad\text{vers}^{-1}\frac{2x}{a}+C.$

22. $\int\frac{-x\,dx}{\sqrt{cx-x^2}}.$ $\qquad\sqrt{cx-x^2}-\frac{c}{2}\text{vers}^{-1}\frac{2x}{c}+C.$

$$\int\frac{-x\,dx}{\sqrt{cx-x^2}}=\int\frac{c-2x-c}{2\sqrt{cx-x^2}}dx=\int\frac{(c-2x)dx}{2\sqrt{cx-x^2}}-\frac{c}{2}\int\frac{dx}{\sqrt{cx-x^2}}.$$

23. $\int\frac{dx}{\sqrt{ax^2-b}}.$ $\qquad\frac{1}{\sqrt{a}}\log(x\sqrt{a}+\sqrt{ax^2-b})+\log c.$

24. $\int\frac{dx}{\sqrt{2ax+x^2}}.$ $\qquad\log(x+a+\sqrt{2ax+x^2})+\log c.$

25. $\displaystyle\int\frac{dx}{x^2+4x}.$ $\qquad$ $\displaystyle\frac{1}{4}\log\frac{x}{x+4}+\log c.$

26. $\displaystyle\int\frac{x\,dx}{(a^4-x^4)^{\frac{1}{2}}}.$ $\qquad$ $\displaystyle\frac{1}{2}\sin^{-1}\frac{x^2}{a^2}+C.$

27. $\displaystyle\int\frac{3\,dx}{4+9x^2}.$ $\qquad$ $\displaystyle\frac{1}{2}\tan^{-1}\frac{3x}{2}+C.$

28. $\displaystyle\int\frac{(b+cx)\,dx}{a^2+x^2}.$ $\qquad$ $\displaystyle\frac{b}{a}\tan^{-1}\frac{x}{a}+\frac{c}{2}\log(a^2+x^2)+C.$

29. $\displaystyle\int\frac{3x^2\,dx}{\sqrt{x^3-9x^6}}.$ $\qquad$ $\displaystyle\frac{1}{3}\operatorname{vers}^{-1}(18x^3)+C.$

30. $\displaystyle\int\frac{dx}{\sqrt{5x^4-3x^2}}.$ $\qquad$ $\displaystyle\frac{1}{3}\sqrt{3}\sec^{-1}\left(x\sqrt{\frac{5}{3}}\right)+C.$

31. $\displaystyle\int\frac{5\,dx}{x\sqrt{3x^2-5}}.$ $\qquad$ $\displaystyle\sqrt{5}\sec^{-1}\left(x\sqrt{\frac{3}{5}}\right)+C.$

32. $\displaystyle\int\frac{(x^2-a^2)^{\frac{1}{2}}}{x}dx.$ $\qquad$ $\displaystyle(x^2-a^2)^{\frac{1}{2}}-a\sec^{-1}\frac{x}{a}+C.$

$$\int\frac{(x^2-a^2)^{\frac{1}{2}}}{x}dx=\int\frac{(x^2-a^2)\,dx}{x\sqrt{x^2-a^2}}=\int\frac{x\,dx}{\sqrt{x^2-a^2}}-\int\frac{a^2\,dx}{x\sqrt{x^2-a^2}}.$$

A fractional differential may often be separated into integrable parts, or reduced to an integrable form, by multiplying its numerator and denominator by the same quantity.

33. $\displaystyle\int\frac{\sqrt{1+x}}{\sqrt{1-x}}dx\left[=\int\frac{(1+x)\,dx}{\sqrt{1-x^2}}\right].$ $\qquad$ $\displaystyle\sin^{-1}x-(1-x^2)^{\frac{1}{2}}+C.$

34. $\displaystyle\int\frac{\sqrt{x+a}}{x\sqrt{x-a}}dx.$ $\qquad$ $\displaystyle\sec^{-1}\frac{x}{a}+\log(x+\sqrt{x^2-a^2})+\log c.$

35. $\displaystyle\int\frac{dx}{(1-x^2)^{\frac{3}{2}}}\left[=\int\frac{x^{-3}\,dx}{(x^{-2}-1)^{\frac{3}{2}}}\right].$ $\qquad$ $\displaystyle\frac{x}{\sqrt{1-x^2}}+C.$

63. Trigonometric Differentials. The following trigonometric differentials are readily reduced to known forms. The forms in the first seven or eight examples should be especially noted.

1. Find $\int\frac{dx}{\sin x}$. *Ans.* $\log\tan(\frac{1}{2}x)+C$.

$$\int\frac{dx}{\sin x}=\int\frac{dx}{2\sin(\frac{1}{2}x)\cos(\frac{1}{2}x)}=\int\frac{\sec^2(\frac{1}{2}x)\frac{1}{2}dx}{\tan(\frac{1}{2}x)}$$
$$=\log\tan(\tfrac{1}{2}x)+C.$$

2. $\int\frac{dx}{\cos x}\left[=\int\frac{dx}{\sin\left(\frac{\pi}{2}+x\right)}\right]$. $\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)+C$.

3. $\int\frac{dx}{\sin x\cos x}\left[=\int\frac{\sec^2 x\,dx}{\tan x}\right]$. $\log\tan x+C$.

4. $\int\cos^2 x\,dx\,[=\int(\frac{1}{2}+\frac{1}{2}\cos 2x)dx]$. $\frac{x}{2}+\frac{1}{4}\sin 2x+C$.

5. $\int\sin^2 x\,dx$. $\frac{x}{2}-\frac{1}{4}\sin 2x+C$.

6. $\int\cot x\,dx\left[=\int\frac{\cos x\,dx}{\sin x}\right]$. $\log\sin x+C$.

7. $\int\tan x\,dx$. $-\log\cos x+C$, or $\log\sec x+C$.

8. $\int\frac{dx}{\sin^2 x\cos^2 x}$. $\tan x-\cot x+C$.

$$\int\frac{dx}{\sin^2 x\cos^2 x}=\int\frac{(\sin^2 x+\cos^2 x)dx}{\sin^2 x\cos^2 x}=\int(\sec^2 x+\operatorname{cosec}^2 x)dx.$$

9. $\int\sin^5 x\,dx$. $-\cos x+\frac{2}{3}\cos^3 x-\frac{\cos^5 x}{5}+C$.

$$\int\sin^5 x\,dx=\int(1-\cos^2 x)^2\sin x\,dx$$
$$=-\int(1-\cos^2 x)^2 d(\cos x).$$

In like manner, $\int \sin^n x\,dx$ and $\int \cos^n x\,dx$ can be found, when n is an odd positive integer.

10. $\int \sin^3 x\,dx.$ $\qquad \frac{1}{3}\cos^3 x - \cos x + C.$

11. $\int \cos^3 x\,dx.$ $\qquad \sin x - \frac{1}{3}\sin^3 x + C.$

12. $\int \cos^5 x\,dx.$ $\qquad \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C.$

13. $\int \sin^5 x \cos^3 x\,dx.$ $\qquad \frac{1}{6}\sin^6 x - \frac{1}{8}\sin^8 x + C.$

$$\int \sin^5 x \cos^3 x\,dx = \int \sin^5 x(1 - \sin^2 x)\cos x\,dx.$$

In like manner, $\int \sin^m x \cos^n x\,dx$ can be found, when either m or n is an odd positive integer.

14. $\int \sin^3 x \cos^3 x\,dx.$ $\qquad \frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C.$

15. $\int \cos^4 x \sin^3 x\,dx.$ $\qquad -\frac{1}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C.$

16. $\int \frac{\cos^3 x\,dx}{\sin^4 x} \left[= \frac{(1 - \sin^2 x)\,d(\sin x)}{\sin^4 x}\right].$ $\qquad \frac{1}{\sin x} - \frac{1}{3\sin^3 x} + C.$

17. $\int \frac{\sin^3 x}{\cos^2 x}dx.$ $\qquad \sec x + \cos x + C.$

18. $\int \frac{\sin^3 x}{\cos^8 x}dx.$ $\qquad \frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x + C.$

19. $\int \frac{\sin^3 x}{\cos^7 x}dx.$ $\qquad \frac{1}{6}\sec^6 x - \frac{1}{4}\sec^4 x + C.$

20. $\int \frac{\sin^2 x}{\cos^6 x}dx.$ $\qquad \frac{1}{5}\tan^5 x + \frac{1}{3}\tan^3 x + C.$

$$\int \frac{\sin^2 x}{\cos^6 x}dx = \int \tan^2 x \sec^4 x\,dx = \int \tan^2 x(\tan^2 x + 1)\sec^2 x\,dx.$$

In like manner, $\frac{\sin^m x}{\cos^n x}dx$ or $\frac{\cos^m x}{\sin^n x}dx$ may be integrated, when $m-n$ is even and negative.

21. $\int\frac{\cos^2 x}{\sin^4 x}dx.$ $\qquad -\frac{1}{3}\cot^3 x + C.$

22. $\int\frac{\sin^5 x}{\cos^7 x}dx.$ $\qquad \frac{1}{6}\tan^6 x + C.$

23. $\int\frac{dx}{\cos^4 x}.$ $\qquad \tan x + \frac{1}{3}\tan^3 x + C.$

24. $\int \tan^2 x\,dx\left[=\int(\sec^2 x - 1)dx\right].$ $\qquad \tan x - x + C.$

25. $\int\cot^2 x\,dx.$ $\qquad -\cot x - x + C.$

26. $\int\tan^3 x\,dx.$ $\qquad \frac{1}{2}\tan^2 x + \log\cos x + C.$

27. $\int\cot^3 x\,dx.$ $\qquad -\frac{1}{2}\cot^2 x - \log\sin x + C.$

28. $\int\tan^5 x\,dx.$

$$\int\tan^5 x\,dx = \int(\sec^2 x - 1)\tan^3 x\,dx = \frac{1}{4}\tan^4 x - \int\tan^3 x\,dx$$
$$= \frac{1}{4}\tan^4 x - \int(\sec^2 x - 1)\tan x\,dx$$
$$= \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \log\cos x + C.$$

In like manner, $\tan^m x\,dx$ and $\cot^m x\,dx$ may be integrated, when m is a whole number.

29. $\int\tan^4 x\,dx.$ $\qquad \frac{1}{3}\tan^3 x - \tan x + x + C.$

30. $\int\cot^4 x\,dx.$ $\qquad -\frac{1}{3}\cot^3 x + \cot x + x + C.$

31. $\int\tan^6 x\,dx.$ $\qquad \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x + C.$

32. $\int \cot^5 x\,dx.$ $\qquad -\frac{1}{4}\cot^4 x + \frac{1}{2}\cot^2 x + \log \sin x + C.$

64. Definite Integrals. All the integrals yet found contain the indeterminate constant term C, and are called *indefinite integrals.*

When C is eliminated, or determined for any hypothesis, the integral is called a *definite integral.*

When, from the data of a problem, we know the value of the integral for some particular value of its variable, C can be determined. For example, suppose that $du = 2\,ax\,dx$, and that $u = 0$ when $x = 2$.

Since $du = 2\,ax\,dx$, $u = ax^2 + C$.

Since $u = 0$ when $x = 2$, $0 = 4\,a + C$.

Hence, $C = -4\,a$, and $u = ax^2 - 4\,a$, a definite integral.

If, in any indefinite integral, two different values of the variable be substituted, and the one result subtracted from the other, C is eliminated, and the integral is said to be taken between *limits.* The symbol for the definite integral of $\phi(x)\,dx$ between the limits a and b is $\int_a^b \phi(x)dx$; a and b are called the limits of integration, a being the *inferior* and b the *superior* limit. The symbol $\int_a^b$ indicates, that the following differential is to be integrated; that a and b are separately to be substituted for the variable in the indefinite integral; and that the first of these results is to be subtracted from the second.

In what precedes, we assume that the integral is continuous between the limits a and b.

In the indefinite integral, neither limit of integration is fixed upon. In the *first* form of the definite integral, only the *inferior* limit is determined, and the integral is still a function of the variable. In the definite integral between limits, both limits are fixed, and the integral ceases to be a function of the variable.

EXAMPLES.

1. Given $dy = (1 + \frac{9}{4}ax)^{\frac{1}{2}}dx$; find the definite integral on the hypothesis that $y = 0$ when $x = 0$.

Here $y=\frac{8}{27a}(1+\frac{9}{4}ax)^{\frac{3}{2}}+C$; $\therefore 0=\frac{8}{27a}+C$.

$$\therefore y=\frac{8}{27a}(1+\tfrac{9}{4}ax)^{\frac{3}{2}}-\frac{8}{27a}.$$

2. Given $dy=(x^3-b^2x)dx$; find the definite integral, if $y=0$ when $x=2$.

Ans. $y=\frac{x^4}{4}-\frac{b^2x^2}{2}+2b^2-4$.

3. Given $dy=\frac{dx}{x}-\frac{dx}{2-x}$; find the definite integral, if $y=0$ when $x=1$.

Ans. $y=\log(2x-x^2)$.

4. Find $\int_a^b nx\,dx$.

Ans. $\frac{n}{2}(b^2-a^2)$.

$$\int nx\,dx=\frac{n}{2}x^2+C;\quad \left[\frac{n}{2}x^2+C\right]_a^{*}=\frac{n}{2}a^2+C;$$

$$\left[\frac{n}{2}x^2+C\right]_b=\frac{n}{2}b^2+C.$$

$$\therefore \int_a^b nx\,dx=\frac{n}{2}b^2+C-\left(\frac{n}{2}a^2+C\right)=\frac{n}{2}(b^2-a^2).$$

5. $\int_0^2 6x^3dx$. 24.

6. $\int_0^a (ax^2-x^3)dx$. $\frac{a^4}{12}$.

7. $\int_0^a \frac{dx}{a^2+x^2}$. $\frac{\pi}{4a}$.

8. $\int_0^a \frac{dx}{\sqrt{a^2-x^2}}$. $\frac{\pi}{2}$.

9. $\int_0^{\frac{1}{4}\pi}\frac{\sin\theta\,d\theta}{\cos^2\theta}\left[=\int_0^{\frac{1}{4}\pi}(\cos\theta)^{-2}\sin\theta\,d\theta\right]$. $\sqrt{2}-1$.

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[*] $\left[\frac{n}{2}x^2+C\right]_a$ denotes the value of $\frac{n}{2}x^2+C$ when $x=a$.

10. $\int_a^b x^n dx.$ $\qquad \frac{b^{n+1}-a^{n+1}}{n+1}.$

11. $\int_0^\infty e^{-ax} dx.$ $\qquad \frac{1}{a}.$

12. $\int_0^{2r} \frac{2\sqrt{2r}\,dy}{\sqrt{2r-y}}.$ $\qquad 8r.$

13. $\int_{-b}^{b} \frac{\pi}{a^4}(y^2-b^2)^4 dy.$ $\qquad \frac{256\pi b^9}{315 a^4}.$

14. $\int_0^\infty \frac{x\,dx}{1+x^4}.$ $\qquad \frac{\pi}{4}.$

15. $\int_0^{\frac{1}{2}\pi} \sin^3 x \cos^3 x\, dx.$ $\qquad \frac{1}{12}.$

Applications to Geometry and Mechanics.

65. Rectification of Curves. From § 16, Cor. 2, we have

$$ds = \sqrt{dx^2 + dy^2};$$

$$\therefore s = \int\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}} dx, \text{ or } \int\left(1+\frac{dx^2}{dy^2}\right)^{\frac{1}{2}} dy.$$

This equation is a general formula for the *rectification* of any plane curve; that is, for finding its length.

Examples.

1. Rectify the semi-cubical parabola $y^2 = ax^3$.

Here $\frac{dy}{dx} = \frac{3ax^2}{2y}$; $\therefore \frac{dy^2}{dx^2} = \frac{9ax}{4}$.

$$\therefore s = \int\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}} dx = \frac{1}{2}\int (4+9ax)^{\frac{1}{2}} dx$$

$$= \frac{(4+9ax)^{\frac{3}{2}}}{27a} + C. \qquad (1)$$

So long as the point from which s is measured is undetermined, C must be indeterminate. If the length of the curve be estimated from the origin, $s = 0$ when $x = 0$.

Substituting these values of s and x in (1), we have

$$0=\frac{8}{27a}+C;\quad \therefore s=\frac{(4+9ax)^{\frac{3}{2}}-8}{27a}. \qquad (2)$$

If, in (2), $a=1$ and $x=\frac{4}{3}$, $s=2\frac{2}{27}$; that is, the arc of $y^2=x^3$ that lies between the origin and $x=\frac{4}{3}$, is $2\frac{2}{27}$ in length.

For the length of the arc, the abscissas of whose extremities are b and c, we have

$$s=\tfrac{1}{2}\int_b^c(4+9ax)^{\frac{1}{2}}dx=\frac{(4+9ac)^{\frac{3}{2}}-(4+9ab)^{\frac{3}{2}}}{27a}.$$

2. Find the length of a branch of the cycloid

$$x=r\operatorname{vers}^{-1}\frac{y}{r}-\sqrt{2ry-y^2}.$$

Here $\dfrac{dx^2}{dy^2}=\dfrac{y}{2r-y}$; $\therefore\left(1+\dfrac{dx^2}{dy^2}\right)^{\frac{1}{2}}=\sqrt{2r}(2r-y)^{-\frac{1}{2}}$.

$$\therefore 2\,\text{AK (Fig. 14)}=2\int_0^{2r}\left(1+\frac{dx^2}{dy^2}\right)^{\frac{1}{2}}dy$$

$$=2\sqrt{2r}\int_0^{2r}(2r-y)^{-\frac{1}{2}}dy=8r.$$

Hence the length of a branch is eight times the radius of the generating circle.

66. Areas of Plane Curves. From § 14, we have

$$dz=ydx;$$

$$\therefore z=\int ydx=\int f(x)dx.$$

This equation is a general formula for finding the area included between any plane curve and the axis of x.

In applying this formula, it must be borne in mind that, area above the axis of x being positive, area below it is negative.

For the area between a curve and the axis of y, we evidently have

$$z=\int xdy.$$

EXAMPLES.

1. Find the area between $y^2 = 2px$ and the axis of x.

Here $z = \int y\,dx = \int (2px)^{\frac{1}{2}}dx = \frac{2}{3}x\sqrt{2px} + C = \frac{2}{3}xy + C.$

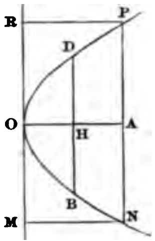

Fig. 15.

If the area be reckoned from the origin, $z = 0$ when $x = 0$;

$\therefore C = 0$, and $z = \frac{2}{3}xy$.

Hence the area OAP $= \frac{2}{3}$ OAPR; and the area of the segment NOP is two-thirds that of the parallelogram MNPR.

If OH $= a$, and OA $= b$,

area BNPD $= 2\int_a^b \sqrt{2px}\,dx = \frac{4}{3}\sqrt{2p}\,(b^{\frac{3}{2}} - a^{\frac{3}{2}})$.

2. Find the area of $y = x^3 + ax^2$ between the limits $x = -a$ and $x = 0$; also between the limits $x = 0$ and $x = a$.

Ans. $\frac{1}{12}a^4$; $\frac{7}{12}a^4$.

3. Find the area of the hyperbola $xy = 1$ between the limits $x = 1$ and $x = a$.

Area $= \log a$; that is, the area is the Naperian logarithm of the superior limit. It is because of this property that Naperian logarithms are sometimes called *hyperbolic logarithms.*

4. Find the area inclosed by the axis of x and the curve $y = x - x^3$.

The inclosed area lies below the axis of x, between $x = -1$ and $x = 0$, and above it, between $x = 0$ and $x = 1$. These two portions being numerically equal, the result obtained by integrating between $x = -1$ and $x = 1$ is 0. To find the required area, obtain the area of each portion separately, and take their numerical sum.

Ans. $\frac{1}{2}$.

5. Find the area of the ellipse $a^2y^2 + b^2x^2 = a^2b^2$.

$$\text{Area} = \frac{b}{a} 4 \int_0^a \sqrt{a^2 - x^2}\, dx. \quad 4 \int_0^a \sqrt{a^2 - x^2}\, dx = \pi a^2;$$

for it evidently equals the area of the circle whose radius is a.

$$\therefore \text{area} = \frac{b}{a} \pi a^2 = \pi ab.$$

6. Find the area intercepted between $y^2 = 2px$ and $x^2 = 2py$.

$$\text{Area} = \int_0^{2p} \sqrt{2px}\, dx - \int_0^{2p} \frac{x^2}{2p}\, dx = \frac{4p^2}{3}.$$

7. Required the area intercepted between $y = \dfrac{x}{1 + x^2}$ and $y = \dfrac{x}{4}$.

Ans. $\log 4 - \frac{6}{8}$.

67. Since $z = \int y\, dx = \int f(x)\, dx$ (§ 66), the integral of $f(x)\, dx$ can be *represented graphically* by the area between the curve $y = f(x)$ and the axis of x. Hence, when $\int f(x)\, dx$ cannot be found, $\int_a^b f(x)\, dx$ can be determined approximately by computing *geometrically* the area of the figure formed by the axis of x, $y = f(x)$, $x = a$, and $x = b$.

68. Areas of Surfaces of Revolution.

Let s represent the length of the curve OPn, and S the surface generated by its revolution about OX as an axis.

To obtain a general formula for the value of S, let $\Delta s =$ arc PP′; then $\Delta S =$ the surface traced by Δs.

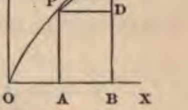

Fig. 16.

Since $$\lim_{\Delta s \doteq 0} \left[\frac{\Delta s}{\text{chord PP}'}\right] = 1; \qquad \S\ 48.$$

$$\therefore \lim_{\Delta s \doteq 0}\left[\frac{\text{surface traced by } \Delta s}{\text{surface traced by PP}'}\right], \text{ or } \lim_{\Delta s \doteq 0}\left[\frac{\dfrac{\Delta S}{\Delta s}}{\dfrac{\text{surface PP}'}{\Delta s}}\right], = 1;$$

$$\therefore \lim_{\Delta s \doteq 0}\left[\frac{\Delta S}{\Delta s}\right] = \lim_{\Delta s \doteq 0}\left[2\pi\left(y + \frac{\Delta y}{2}\right)\frac{\text{chord PP}'}{\Delta s}\right].$$

$$\therefore \frac{dS}{ds} = 2\pi y.$$

$$\therefore S = 2\pi\int y\,ds = 2\pi\int y\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}}dx.$$

When the axis of revolution is the axis of y, we have, similarly,

$$S = 2\pi\int x\,ds = 2\pi\int x\left(1 + \frac{dx^2}{dy^2}\right)^{\frac{1}{2}}dy.$$

EXAMPLES.

1. Find the surface of the sphere.

Here the generating curve is $x^2 + y^2 = r^2$.

$$\therefore S = 2\pi\int_{-r}^{r} y\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}}dx = 2\pi\int_{-r}^{r} y\left(1 + \frac{x^2}{y^2}\right)^{\frac{1}{2}}dx$$

$$= 2\pi\int_{-r}^{r} r\,dx = 4\pi r^2.$$

2. Find the surface of the paraboloid; that is, of the surface traced by the revolution of a parabola about its axis.

$$S = 2\pi\int_0^b y(dx^2 + dy^2)^{\frac{1}{2}} = 2\pi\int_0^b y\left(1 + \frac{dx^2}{dy^2}\right)^{\frac{1}{2}}dy$$

$$= \frac{2\pi}{3p}\left[(b^2 + p^2)^{\frac{3}{2}} - p^3\right].$$

69. Volumes of Solids of Revolution.

Let V represent the volume generated by the revolution of OPn about OX as an axis. To deduce a general formula for the value of V, let $\Delta x = \text{AB}$; then $\Delta y = \text{DP}'$, and $\Delta V =$ the volume generated by the revolution of ABP′P.

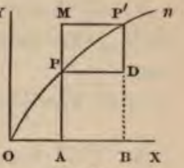

Fig. 17.

Now, volume ABDP $< \Delta V <$ volume ABP′M;

or $$\pi y^2 \Delta x < \Delta V < \pi (y + \Delta y)^2 \Delta x;$$

$$\therefore \pi y^2 < \frac{\Delta V}{\Delta x} < \pi (y + \Delta y)^2;$$

$$\therefore \frac{dV}{dx} = \pi y^2, \text{ or } V = \pi \int y^2 dx. \qquad (1)$$

Or, to obtain (1), conceive the solid as generated by a variable circle, whose centre moves along the axis of the solid, and whose radius is equal to the ordinate of the generating curve. With this conception, it is evident that, if $dx = \text{AB}$, $dV =$ the cylinder whose altitude is AB, and the radius of whose base is AP.

$$\therefore dV = \pi y^2 dx, \text{ or } V = \pi \int y^2 dx.$$

When the axis of revolution is the axis of y, we have, similarly,

$$V = \pi \int x^2 dy.$$

EXAMPLES.

1. Find the volume of the prolate spheroid; that is, of the solid generated by the ellipse revolving about its major axis.

Here $V = \pi \int_{-a}^{a} y^2 dx = \pi \int_{-a}^{a} \frac{b^2}{a^2}(a^2 - x^2) dx = \frac{2}{3}(2 a\pi b^2).$

Hence the volume of the prolate spheroid is two-thirds the volume of its circumscribed cylinder of revolution.

If $a = b$, $V = \frac{4}{3}\pi a^3$,

which gives the volume of the sphere whose radius is a.

2. Find the volume generated by the revolution of $y^3 = cx$ about the axis of x, volume being measured from the origin.

Ans. $V = \frac{3}{5}\pi y^2 x = \frac{3}{5}$ the circumscribed cylinder.

3. Find the volume of the oblate spheroid; that is, of the solid generated by the revolution of the ellipse about its minor axis.

Ans. $V = \frac{4}{3}\pi a^2 b = \frac{2}{3}$ the circumscribed cylinder of revolution.

4. Find the volume of the paraboloid; that is, of the solid generated by the revolution of the parabola about its axis.

Ans. $V = \frac{1}{2}\pi x y^2$, or $\frac{1}{2}$ the circumscribed cylinder.

5. Find the inclosed volume of the solid generated by the revolution of $y^2 - b^2 = ax^{\frac{1}{4}}$ about the axis of y.

Ans. $V = \frac{\pi}{a^4}\int_{-b}^{b}(y^2 - b^2)^4 dy = \frac{256\pi b^9}{315 a^4}$.

70. From $v = \frac{ds}{dt}$ and $a = \frac{dv}{dt}$ (§ 33), we have the following **Fundamental Formulas of Mechanics.**

$$\text{I.} \quad v = \frac{ds}{dt}; \therefore s = \int v\,dt, \text{ and } t = \int \frac{ds}{v}.$$

$$\text{II.} \quad a = \frac{dv}{dt}; \therefore v = \int a\,dt, \text{ and } t = \int \frac{dv}{a}.$$

EXAMPLES.

1. The acceleration of a moving body is constant; find the velocity and the distance.

$$v = \int a\,dt = at + C = at + v_0, \tag{1}$$

in which v_0 represents the *initial* velocity; that is, the value of v when $t = 0$.

$$s = \int v\,dt = \int (at + v_0)\,dt = \tfrac{1}{2}at^2 + v_0 t + s_0, \tag{2}$$

in which s_0 represents the *initial* distance; that is, the value of s when $t = 0$.

If the motion begins when $t=0$, $v_0=0$ and $s_0=0$; hence (1) and (2) become

$$v=at \quad \text{and} \quad s=\tfrac{1}{2}at^2;$$

$$\therefore t=\sqrt{\frac{2s}{a}} \text{ and } v=\sqrt{2as}.$$

These four formulas are the *fundamental formulas* for *uniformly accelerated* motion.

The acceleration caused by gravity is 32.17+ ft. a second, and is denoted by g. If we substitute g for a in the four formulas given above, we obtain the formulas for the free fall of bodies in vacuo near the earth's surface.

2. By a principle of Mechanics, if AB be a vertical line, the acceleration of a body sliding without friction along the inclined plane AC is $g\cos\phi$, in which $\phi=$ angle BAC. Let s', v', and t' represent respectively the distance AC, the velocity acquired along AC, and the time of descent; then, from the formulas,

$$v=\sqrt{2as}, \qquad \text{and } t=\sqrt{\frac{2s}{a}},$$

we have

$$v'=\sqrt{2gs'\cos\phi}, \text{ and } t'=\sqrt{\frac{2s'}{g\cos\phi}}.$$

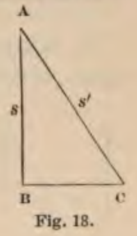

Fig. 18.

Let s represent the vertical distance AB; then

$$s'=\frac{s}{\cos\phi}, \qquad v'=\sqrt{2gs'\cos\phi}=\sqrt{2gs},$$

and $$t'=\sqrt{\frac{2s}{g\cos^2\phi}}=\frac{1}{\cos\phi}\sqrt{\frac{2s}{g}}.$$

Hence the velocity acquired by a body sliding without friction down AC equals that acquired by a body falling vertically down AB; and the time of descent along AC is $\frac{1}{\cos\phi}$ the time of descent along AB.

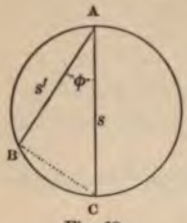

Fig. 19.

3. Let AC ($=s$) be the vertical diameter of any vertical circle ABC; then the time of descent from A along any chord AB ($=s'$) is

$$\sqrt{\frac{2s'}{g\cos\phi}} \text{ (Ex. 2), which equals } \sqrt{\frac{2s}{g}},$$

since $\cos\phi = \frac{s'}{s}$.

Hence the time of descent from A along any chord of the circle is the same as that along the vertical diameter.

4. The acceleration varies directly as the time from a state of rest of the body; that is, $\frac{dv}{dt} = a = ct$; find v and s at the end of time t.

Ans. $v = \frac{1}{2}ct^2$; $s = \frac{1}{6}ct^3$.

5. When the velocity is a given function of the time, the time, velocity, distance, and acceleration can be represented geometrically, as follows:

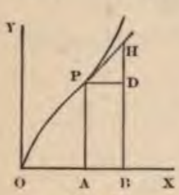

Fig. 20.

Construct the locus of $v = f(t)$, t being represented by abscissas, and v by ordinates, the unit of t being represented by the same unit of length as the unit of v. Then, by § 66, the area between the curve and the axis of abscissas equals $\int v\,dt$; but, by § 70, $s = \int v\,dt$.

Hence, if abscissas represent time, and ordinates velocities, the area between the curve $v = f(t)$ and the axis of abscissas represents the distance traversed by the moving body.

Again, if PH is a tangent at P, and AB represents the unit of time, DH represents the acceleration at the end of the second unit of time; for it represents what would be the increase of the velocity, or ordinate, in a unit of time, if this increase became uniform at the end of the second unit of time.

6. A body starts from O (Fig. 21); its velocity in the direction of OY is constant, and in the direction of OX is gt; what is its path?

Let OX and OY be the axes of x and y, respectively;

then $$\frac{dy}{dt}=c, \text{ and } \frac{dx}{dt}=gt.$$

Hence, $y=ct$, and $x=\frac{g}{2}t^2$;

$$\therefore y^2=\frac{2c^2}{g}x. \qquad (1)$$

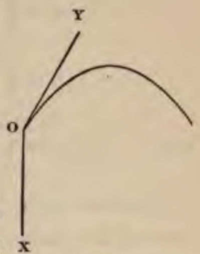

Fig. 21.

Since (1) is the equation of a parabola referred to a diameter and a tangent at its vertex, the path of the body is an arc of a parabola. Hence, if it were not for the resistance of the atmosphere, the path of a projectile, as a ball from a rifle, would be an arc of a parabola; for its velocity would be gt along the action-line of gravity, and constant along the line of projection.

7. The velocity of a body in the direction of OX is $12t$, and in the direction of OY is $4t^2-9$; find the velocity along its path onm, the accelerations and distances in the direction of each axis and along the line of its path, and the equation of its path.

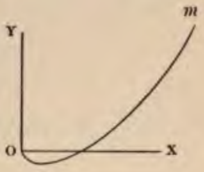

Fig. 22.

Let v_x, v_y, v_s and a_x, a_y, a_s represent respectively the velocities and accelerations in the directions of the axes of x and y and along the path, whose length we will represent by s.

Then $$\frac{dx}{dt}=v_x=12t, \text{ and } \frac{dy}{dt}=v_y=4t^2-9.$$

$$\therefore v_s=\frac{ds}{dt}=\sqrt{\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}}=\sqrt{144t^2+(4t^2-9)^2}$$

$$=\sqrt{16t^4+72t^2+81}=4t^2+9.$$

The accelerations are

$$a_x=\frac{dv_x}{dt}=12; \quad a_y=\frac{dv_y}{dt}=8t; \text{ and } a_s=\frac{dv_s}{dt}=8t.$$

The distances are

$$x = \int 12\,t\,dt = 6\,t^2, \tag{1}$$

$$y = \int (4t^2 - 9)\,dt = \tfrac{4}{3}t^3 - 9\,t, \tag{2}$$

and $$s = \int (4\,t^2 + 9)\,dt = \tfrac{4}{3}t^3 + 9\,t.$$

Eliminating t between (1) and (2), we obtain for the equation of the path,

$$y = (\tfrac{2}{9}x - 9)\sqrt{\frac{x}{6}}.$$

The form of the path is shown in Fig. 22. (See Weisbach's Mechanics of Engineering, page 148.)

CHAPTER IV.

SUCCESSIVE DIFFERENTIATION.

71. Successive Derivatives. Since $f'(x)$, the derivative of $f(x)$, is in general a function of x, it can be differentiated. The derivative of $f'(x)$ is called the *second derivative* of the original function $f(x)$, and is denoted by $f''(x)$. The derivative of $f''(x)$ is called the *third* derivative of $f(x)$, and is represented by $f'''(x)$; and so on. $f^n(x)$ represents the n*th derivative* of $f(x)$, or the derivative of $f^{n-1}(x)$.

Thus, if

$$f(x) = x^4; \qquad f'(x) = \frac{d}{dx}(x^4) = 4x^3;$$

$$f''(x) = \frac{d}{dx}(4x^3) = 12x^2; \qquad f'''(x) = \frac{d}{dx}(12x^2) = 24x;$$

$$f^{\text{IV}}(x) = 24; \qquad f^{\text{V}}(x) = 0.$$

$f'(x)$, $f''(x)$, $f'''(x)$, etc., are the successive derivatives of $f(x)$.

72. Signification of $\mathbf{f^n(x)}$. Since $f^n(x)$ is obtained from $f^{n-1}(x)$ in the same way that $f'(x)$ is from $f(x)$, *the* n*th derivative of a function expresses the ratio of the rate of change of its* (n − 1)*th derivative to that of its variable; and the* (n − 1)*th derivative is an increasing or a decreasing function, according as the* n*th derivative is positive or negative.*

Cor. If a is finite, and $f(a)$* $= \infty$, $f'(a) = \infty$, $f''(a) = \infty$, etc.

For, when $f(a) = \infty$, $f(a + h)$ is not ∞, however small h be taken. Hence, while x changes a very small amount from a,

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* $f(a)$ represents the value of $f(x)$ for $x = a$. The equation $f(a) = \infty$ means that $f(x)$ increases without limit, as x approaches a as its limit.

$f(x)$ changes an infinite amount. Therefore, when $x=a$, $f(x)$ must change infinitely faster than x does; hence $f'(a)=\infty$. For like reason, if $f'(a)=\infty$, $f''(a)=\infty$, etc.

EXAMPLES.

1. Find the successive derivatives of x^3+2x^2+x+7.

Let $f(x)=x^3+2x^2+x+7$;

then $$f'(x)=\frac{d}{dx}(x^3+2x^2+x+7)=3x^2+4x+1;$$

$$f''(x)=\frac{d}{dx}(3x^2+4x+1)=6x+4;$$

$$f'''(x)=\frac{d}{dx}(6x+4)=6;$$

and $$f^{\text{IV}}(x)=0.$$

2. Find the successive derivatives of cx^3+ax^2+a.

3. If $f(x)=x^3\log x$, prove that $f^{\text{IV}}(x)=\frac{6}{x}$.

4. If $f(x)=e^{ax}$, prove that $f^n(x)=a^n e^{ax}$.

$f^n(x)$ is written out in accordance with the law discovered by inspecting $f''(x)$ and $f'''(x)$.

5. If $f(x)=\sin mx$, prove that $f^{\text{IV}}(x)=m^4\sin mx$.

6. If $f(x)=x^4\log x$, prove that $f^{\text{VI}}(x)=\frac{-\underline{|4}}{x^2}$.*

7. If $f(x)=x^x$, prove that $f''(x)=x^x(1+\log x)^2+x^{x-1}$.

8. If $f(x)=\tan x$, prove that $f'''(x)=6\sec^4 x-4\sec^2 x$.

9. If $f(x)=\log(e^x+e^{-x})$, prove that $f'''(x)=-8\frac{e^x-e^{-x}}{(e^x+e^{-x})^3}$.

10. If $f(x)=\frac{x^3}{1-x}$, prove that $f^{\text{IV}}(x)=\frac{24}{(1-x)^5}$.

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* $\underline{|n}$, read "factorial n," stands for $1\times2\times3\times4\times\cdots\times n$.

11. If $f(x)=a^x$, prove that $f^n(x)=(\log a)^n a^x$.

12. If $f(x)=\log(1+x)$, prove that $f^n(x)=\dfrac{(-1)^{n-1}\underline{|n-1}}{(1+x)^n}$.

$$f'(x)=\frac{1}{1+x};\ f''(x)=\frac{(-1)}{(1+x)^2};\ f'''(x)=\frac{(-1)^2\underline{|2}}{(1+x)^3}.$$

$$f^{\text{IV}}(x)=\frac{(-1)^3\underline{|3}}{(1+x)^4};\ \therefore f^n(x)=\frac{(-1)^{n-1}\underline{|n-1}}{(1+x)^n}.$$

13. If $f(x)=(1+x)^m$, prove that

$$f^n(x)=m(m-1)\cdots(m-n+1)(1+x)^{m-n}.$$

73. Successive Differentials. The differential of the first differential of a function is called the *second differential* of the function. The differential of the second differential is called the *third differential.* In like manner, we have the *fourth*, *fifth*, and n*th* differentials. $d(dy)$ is written d^2y, and read "second differential of y"; $d(d^2y)$ is written d^3y, and read "third differential of y"; and so on.

dy, d^2y, d^3y, etc., are the *successive differentials* of y.

In differentiating $y=f(x)$ successively, dx is usually regarded as constant; that is, as having the same value for all values of x. This greatly simplifies the second and higher differentials, and also the relations between the successive differentials and derivatives, and is allowable; for, when independent, x may be regarded as changing uniformly.

74. Relations between the Successive Differentials and Derivatives.

If $\quad y=f(x),\ \dfrac{dy}{dx}=f'(x);\ \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)=f''(x)$; etc.

If dx is variable, $\dfrac{dy}{dx}$ is a fraction with a variable numerator and denominator, and we have

$$f''(x)=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{dx\,d^2y-dy\,d^2x}{dx^3}. \tag{1}$$

But, if dx is constant, $\frac{dy}{dx}$ is the product of the constant $\frac{1}{dx}$ and the variable dy, and we have

$$f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}. \tag{2}$$

For the same hypothesis we have

$$f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3};\ f^{\text{IV}}(x) = \frac{d^4y}{dx^4};\ f^n(x) = \frac{d^ny}{dx^n}.$$

Hence $d^ny = f^n(x)dx^n$; whence $f^n(x)$ is often called the n*th* *differential coefficient* of $f(x)$.

For the hypothesis that dx is constant, $d^2x = 0$, and equation (1) becomes (2), as it evidently should.

Examples.

1. Find the successive differentials of x^4.

Let $y = x^4$; then $dy = 4x^3dx$.

Differentiating this last equation, regarding dx as constant, we have

$$d^2y = (4\,dx)\,d(x^3) = 12\,x^2dx^2;$$
$$\therefore d^3y = (12\,dx^2)\,d(x^2) = 24\,x\,dx^3;$$
$$\therefore d^4y = 24\,dx^4;\ \therefore d^5y = 0.$$

2. Find the successive differentials of $5x^3 + 2x^2 - 3x$.

If $y = 5x^3 + 2x^2 - 3x$, $d^3y = 30\,dx^3$.

3. If $y = \sin x$, prove that $d^4y = \sin x\,dx^4$.

4. If $y = \log(ax)$, prove that $d^4y = -\frac{6}{x^4}dx^4$.

5. If $y = 2a\sqrt{x}$, prove that $\frac{d^3y}{dx^3} = \frac{3a}{4x^{\frac{5}{2}}}$.

6. If $y = \log\sin x$, prove that $\frac{d^3y}{dx^3} = \frac{2\cos x}{\sin^3 x}$.

7. If $y = x^2 \log x$, prove that $\dfrac{d^3y}{dx^3} = \dfrac{2}{x}$.

8. If $y = \cos mx$, prove that $\dfrac{d^4y}{dx^4} = m^4 \cos mx$.

9. If $y^2 = 2px$, prove that $\dfrac{d^3y}{dx^3} = \dfrac{3p^3}{y^5} = \dfrac{3p^{\frac{1}{2}}}{(2x)^{\frac{5}{2}}}$.

$$\frac{dy}{dx} = \frac{p}{y}; \quad \therefore \frac{d^2y}{dx^2} = \frac{-p\dfrac{dy}{dx}}{y^2} = -\frac{p^2}{y^3};$$

$$\therefore \frac{d^3y}{dx^3} = \frac{3y^2p^2\dfrac{dy}{dx}}{y^6} = \frac{3p^3}{y^5} = \frac{3p^{\frac{1}{2}}}{(2x)^{\frac{5}{2}}}.$$

10. If $a^2y^2 + b^2x^2 = a^2b^2$, prove that $\dfrac{d^2y}{dx^2} = -\dfrac{b^4}{a^2y^3}$.

11. If $x^2 + y^2 = r^2$, prove that $\dfrac{d^2y}{dx^2} = -\dfrac{r^2}{y^3}$.

12. If $y^2 = \sec 2x$, prove that $y + \dfrac{d^2y}{dx^2} = 3y^5$.

13. If $y = e^x \sin x$, prove that $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 0$.

14. Given $s = 4t^3$; find v the velocity, and a the acceleration.

Since $v = \dfrac{ds}{dt}$, $a = \dfrac{dv}{dt} = \dfrac{d}{dt}\left(\dfrac{ds}{dt}\right) = \dfrac{d^2s}{dt^2}$.

Hence $v = \dfrac{ds}{dt} = 12t^2$, and $a = \dfrac{d^2s}{dt^2} = 24t$.

15. If $s = ct^2 + bt$, what is the velocity and acceleration?

Ans. $v = 2ct + b$; $a = \dfrac{d^2s}{dt^2} = 2c$.

16. If $y^2 = 2px$, prove that $\dfrac{d^2x}{dy^2} = \dfrac{1}{p}$.

Here x is regarded as the function, and dy is constant.

CHAPTER V.

SUCCESSIVE INTEGRATION, AND APPLICATIONS.

75. The general formulas for integration enable us to obtain the original function from which a second, third, fourth, or *n*th differential has been derived.

For example, let $d^3y = 5\,b\,dx^3$; then

$$d\left(\frac{d^2y}{dx^2}\right) = 5\,b\,dx. \tag{1}$$

Integrating (1), we have

$$\frac{d^2y}{dx^2} = 5\,bx + C_1; \quad \therefore d\left(\frac{dy}{dx}\right) = 5\,bx\,dx + C_1\,dx;$$

$$\therefore \frac{dy}{dx} = \tfrac{5}{2}bx^2 + C_1x + C_2;$$

$$\therefore y = \tfrac{5}{6}bx^3 + \tfrac{1}{2}C_1x^2 + C_2x + C_3.$$

EXAMPLES.

1. Given $d^3y = 0$, to find y.

Here $\quad \dfrac{d^3y}{dx^3} = 0; \quad \therefore d\left(\dfrac{d^2y}{dx^2}\right) = 0; \quad \therefore \dfrac{d^2y}{dx^2} = C_1;$

$$\therefore d\left(\frac{dy}{dx}\right) = C_1\,dx; \quad \therefore \frac{dy}{dx} = C_1x + C_2;$$

$$\therefore y = \tfrac{1}{2}C_1x^2 + C_2x + C_3.$$

2. Given $d^4y = \sin x\,dx^4$, to find y.

$$d\left(\frac{d^3y}{dx^3}\right) = \sin x\,dx; \quad \therefore \frac{d^3y}{dx^3} = -\cos x + C_1; \text{ etc.}$$

Ans. $y = \sin x + \frac{1}{6}C_1x^3 + \frac{1}{2}C_2x^2 + C_3x + C_4$.

3. Given $\frac{d^3y}{dx^3} = 3x^2 - x^{-3}$, to find y.

$$y = \tfrac{1}{20}x^5 - \tfrac{1}{2}\log x + \tfrac{1}{2}C_1x^2 + C_2x + C_3.$$

4. Given $\frac{d^3y}{dx^3} = \sin x$, to find y.

$$y = \cos x + \tfrac{1}{2}C_1x^2 + C_2x + C_3.$$

5. Given $d^3y = 2x^{-3}dx^3$, to find y.

$$y = \log x + \tfrac{1}{2}C_1x^2 + C_2x + C_3.$$

76. Problems in Mechanics.

1. If the acceleration of a body moving toward a centre of force varies directly as its distance from that centre, determine the velocity and time.

Let $\mu =$ the acceleration at a unit's distance from the centre of force;

$x =$ the *varying*, and c the *initial*, distance of the body from that centre;

then $x\mu =$ the acceleration at the distance x.

Here $$s = c - x; \quad \therefore v = \frac{ds}{dt} = -\frac{dx}{dt}; \tag{1}$$

and $$x\mu = a = \frac{d^2s}{dt^2} = -\frac{d^2x}{dt^2}. \tag{2}$$

Multiplying (2) by $-dx$, we obtain

$$\frac{dx}{dt}d\left(\frac{dx}{dt}\right) = -\mu x\,dx;$$

$$\therefore \left(\frac{dx}{dt}\right)^2 [= v^2] = -\mu x^2 + C.$$

Since $v = 0$ when $x = c$, $C = \mu c^2$;

$$\therefore v = -\frac{dx}{dt} = \sqrt{\mu(c^2 - x^2)}\,; \tag{3}$$

$$\therefore t = \mu^{-\frac{1}{2}}\cos^{-1}\frac{x}{c}. \tag{4}$$

$C=0$, since $t=0$ when $x=c$.

If in (3) and (4) we make $x=0$, we have

$$v=c\sqrt{\mu}, \text{ the velocity at the centre of force;}$$

and $$t=\tfrac{1}{2}\pi\mu^{-\frac{1}{2}},\ \tfrac{3}{2}\pi\mu^{-\frac{1}{2}}, \text{ etc.}$$

Hence the time required for the body to reach the centre of force is independent of its initial distance from that centre.

Below the surface of the earth, the acceleration due to gravity varies as the distance from its centre. Hence, from (3) we learn that if a body could pass freely through the earth, it would fall with an increasing velocity from the surface to the centre, from which it would move on with a decreasing velocity, until it reached the surface on the opposite side. It would then return to its first position, and thus move to and fro.

The acceleration due to gravity at the surface of the earth being g, and the radius being r, we have in this case,

$$\mu=\frac{g}{r};$$

$$\therefore v=r\sqrt{\mu}=\sqrt{gr}, \text{ the velocity at the centre;}$$

and $$2t=\pi\mu^{-\frac{1}{2}}=\pi\sqrt{\frac{r}{g}}=3.1416\sqrt{\frac{20919360}{32.17}}\text{ sec.}$$

$$=42 \text{ min. } 13.4 \text{ sec.,}$$

which is the time that would be required for a body to fall through the earth.

2. Assuming that the acceleration of a falling body above the surface of the earth varies inversely as the square of its distance from the earth's centre, find the velocity and time.

Let $x=$ the *varying*, and c the *initial*, distance of the body from the earth's centre;

$r=$ the radius of the earth;

$g=$ the acceleration due to gravity at its surface;

$a=$ the acceleration due to gravity at the distance x.

Then $s = c - x$; and, from the law of fall, we have

$$a : g :: r^2 : x^2;$$

$$\therefore a = \frac{gr^2}{x^2} = -\frac{d^2x}{dt^2}. \tag{1}$$

Multiplying (1) by dx, and integrating, we have

$$v = -\frac{dx}{dt} = (2gr^2)^{\frac{1}{2}}\left(\frac{1}{x} - \frac{1}{c}\right)^{\frac{1}{2}}. \tag{2}$$

If $c = \infty$; that is, if the body fall from an infinite distance to the earth, we have from (2), when $x = r$,

$$v = \sqrt{2gr}.$$

Since $g = 32\frac{1}{6}$ ft., and $r = 3962$ miles, we have

$$v = \left(\frac{64\frac{1}{3}}{5280} \times 3962\right)^{\frac{1}{2}} = 6.95 + \text{ miles.}$$

Hence, the maximum velocity with which a falling body can reach the earth is less than seven miles per second.

From (2), we have

$$\frac{dx}{dt} = -(2gr^2)^{\frac{1}{2}}\left(\frac{cx - x^2}{cx^2}\right)^{\frac{1}{2}} = -\left(\frac{2gr^2}{c}\right)^{\frac{1}{2}}\frac{(cx - x^2)^{\frac{1}{2}}}{x};$$

$$\therefore dt = \left(\frac{c}{2gr^2}\right)^{\frac{1}{2}}\frac{-x\,dx}{\sqrt{cx - x^2}} = \left(\frac{c}{2gr^2}\right)^{\frac{1}{2}}\frac{c - 2x - c}{2\sqrt{cx - x^2}}dx;$$

$$\therefore\ t = \left(\frac{c}{2gr^2}\right)^{\frac{1}{2}}\left[(cx - x^2)^{\frac{1}{2}} - \frac{c}{2}\text{vers}^{-1}\frac{2x}{c}\right] + C.$$

Since $t = 0$ when $x = c$, $C = \frac{1}{2}c\pi\left(\frac{c}{2gr^2}\right)^{\frac{1}{2}}$;

$$\therefore\ t = \left(\frac{c}{2gr^2}\right)^{\frac{1}{2}}\left[(cx - x^2)^{\frac{1}{2}} - \frac{c}{2}\text{vers}^{-1}\frac{2x}{c} + \tfrac{1}{2}c\pi\right]. \tag{3}$$

3. Assuming that r, the radius of the earth, is 3962 miles; that the sun is $24{,}000\,r$ distant from the earth; and that the moon is $60\,r$ distant; find the time that it would take a body to fall from the moon to the earth, and the velocity, at the earth's

surface, of a body falling from the sun. The attraction of the moon and sun, and the resistance of any medium, are not to be considered.

4. A body falls in the air by the force of gravity; the resistance of the air varying as the square of the velocity, determine the velocity on the hypothesis that the force of gravity is constant.

Let $\mu =$ the resistance when the velocity is unity;

and $t =$ the time of falling through the distance s.

Then $\mu\left(\frac{ds}{dt}\right)^2 =$ the resistance of the air for any velocity;

and $g =$ the acceleration downward due to gravity alone.

Hence $g - \mu\left(\frac{ds}{dt}\right)^2 =$ the actual acceleration downward;

that is, $$\frac{d^2s}{dt^2} = g - \mu\left(\frac{ds}{dt}\right)^2; \tag{1}$$

$$\therefore \frac{d\left(\frac{ds}{dt}\right)}{\mu\, dt} = \frac{g}{\mu} - \left(\frac{ds}{dt}\right)^2, \text{ or } \mu\, dt = \frac{d\left(\frac{ds}{dt}\right)}{\frac{g}{\mu} - \left(\frac{ds}{dt}\right)^2}. \tag{2}$$

Observing that the second member of (2) is of the form $\frac{dx}{a^2 - x^2}$, and integrating, we have

$$\mu t = \frac{1}{2}\left(\frac{\mu}{g}\right)^{\frac{1}{2}} \log \frac{g^{\frac{1}{2}} + \mu^{\frac{1}{2}}\frac{ds}{dt}}{g^{\frac{1}{2}} - \mu^{\frac{1}{2}}\frac{ds}{dt}}. \tag{3}$$

$$C = 0; \text{ since } t = 0 \text{ when } \frac{ds}{dt}\,[= v] = 0.$$

From (3), by principles of logarithms, we obtain

$$\frac{g^{\frac{1}{2}} + \mu^{\frac{1}{2}}\frac{ds}{dt}}{g^{\frac{1}{2}} - \mu^{\frac{1}{2}}\frac{ds}{dt}} = e^{2t\sqrt{\mu g}};$$

$$\therefore \frac{ds}{dt}=\left(\frac{g}{\mu}\right)^{\frac{1}{2}}\frac{e^{2t\sqrt{\mu g}}-1}{e^{2t\sqrt{\mu g}}+1}=v.$$

Hence, as t increases, v rapidly approaches the constant value $\left(\frac{g}{\mu}\right)^{\frac{1}{2}}$.

5. A body is projected with a velocity v_0 into a medium which resists as the square of the velocity; determine the velocity and distance after t seconds.

Let $\mu =$ the resistance of the medium when the velocity is unity;

and $s =$ the distance passed over in t seconds.

Then $\mu\left(\frac{ds}{dt}\right)^2 =$ the resistance for any velocity.

Hence $$\frac{d^2s}{dt^2}=-\mu\left(\frac{ds}{dt}\right)^2, \quad \text{or} \quad \frac{d\left(\frac{ds}{dt}\right)}{\frac{ds}{dt}}=-\mu\, ds;$$

$$\therefore \log\frac{ds}{dt}=-\mu s+C=-\mu s+\log v_0.$$

$C=\log v_0$, since $\frac{ds}{dt}=v_0$ when $s=0$.

Hence $$-\mu s=\log\frac{ds}{dt}-\log v_0=\log\left(\frac{ds}{dt}\div v_0\right);$$

$$\therefore \frac{ds}{dt}\div v_0=e^{-\mu s}, \quad \text{or} \quad \frac{ds}{dt}=\frac{v_0}{e^{\mu s}}. \tag{1}$$

Hence, the velocity decreases rapidly, but becomes zero only when $s=\infty$.

Integrating (1), and solving the resulting equation for s, we obtain

$$s=\frac{1}{\mu}\log(\mu v_0 t+1).$$

6. A body slides without friction down a given curve; required the velocity it acquires under the influence of gravity.

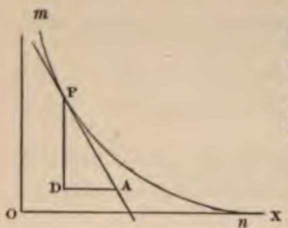

Fig. 23.

Let mn be the given curve, PA a tangent at any point P, and $\text{PA} = ds$; then $-\text{PD} = dy$, and the acceleration caused by gravity at P equals $g \cos \phi$, in which $\phi = \text{DPA}$ (§ 70, Ex. 2).

$$\therefore \frac{d^2s}{dt^2} = g \cos \phi = -g \frac{dy}{ds};$$

$$\therefore 2 \frac{ds}{dt} d\left(\frac{ds}{dt}\right) = -2g\,dy;$$

$$\therefore \left(\frac{ds}{dt}\right)^2 [= v^2] = -2gy + C.$$

If y_0 be the ordinate of the starting-point on the curve, $v = 0$ when $y = y_0$, and $C = 2gy_0$.

$$\therefore v = \sqrt{2g(y_0 - y)}.$$

When $y = 0$, $v = \sqrt{2gy_0}$, which is the velocity that the body would acquire in falling the vertical distance y_0 (§ 70, Ex. 1). Hence, whatever be the curve down which, from any point P, a body slides without friction, it has the same velocity when it reaches the line OX.

7. The base of a cycloid is horizontal, and its vertex is downward; find the time of descent of a heavy body from any point on the curve to its vertex.

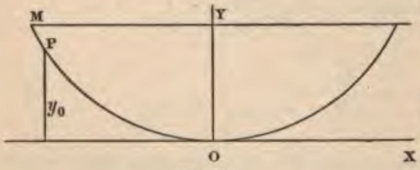

Fig. 24.

Let the vertex O be the origin of coördinates, y_0 the ordinate of the starting-point P, and s the length of the curve reckoned from that point. Then, from the previous problem, we have

$$v = \sqrt{2g(y_0 - y)} = \frac{ds}{dt}; \therefore dt = \frac{ds}{\sqrt{2g(y_0 - y)}}. \quad (1)$$

Since ds is positive, and dy negative,

$$ds = -\left(\frac{dx^2}{dy^2}+1\right)^{\frac{1}{2}}dy.$$

The equation of the cycloid referred to the axes OX and OY is

$$x = r\,\mathrm{vers}^{-1}\frac{y}{r} + \sqrt{2ry-y^2}.$$

$$\therefore \frac{dx}{dy} = \left(\frac{2r-y}{y}\right)^{\frac{1}{2}};$$

$$\therefore ds = -\left(\frac{dx^2}{dy^2}+1\right)^{\frac{1}{2}}dy = -\left(\frac{2r}{y}\right)^{\frac{1}{2}}dy. \qquad (2)$$

From equations (1) and (2), we have

$$dt = -\left(\frac{2r}{y}\right)^{\frac{1}{2}}\frac{dy}{\sqrt{2g(y_0-y)}} = -\left(\frac{r}{g}\right)^{\frac{1}{2}}\frac{dy}{\sqrt{y_0y-y^2}}.$$

$$\therefore\ t = -\left(\frac{r}{g}\right)^{\frac{1}{2}}\mathrm{vers}^{-1}\frac{2y}{y_0} + C.$$

Since $t = 0$ when $y = y_0$,

$$C = \left(\frac{r}{g}\right)^{\frac{1}{2}}\mathrm{vers}^{-1}2 = \pi\left(\frac{r}{g}\right)^{\frac{1}{2}}.$$

$$\therefore\ t = \left(\frac{r}{g}\right)^{\frac{1}{2}}\left(\pi - \mathrm{vers}^{-1}\frac{2y}{y_0}\right);$$

$$\therefore\ t = \pi\left(\frac{r}{g}\right)^{\frac{1}{2}},\ \text{when } y = 0.$$

Hence, the time required to reach the lowest point O will be the same, from whatever point on OM the body starts. Hence, if a pendulum swings in the arc of a cycloid, the time required for one oscillation is $2\pi\sqrt{\frac{r}{g}}$. The time of an oscillation being independent of the length of the arc, the cycloidal pendulum is isochronal.

Cor. To find the time of descent along any other curve, we would obtain from its equation the value of ds, substitute this in equation (1), and integrate between the proper limits.

8. To find the length and equation of the *Catenary.*

Let NOM represent the form assumed by a chain, or perfectly flexible cord, of uniform section and density, when suspended from any two fixed points M and N; then is NOM a *catenary.* Let O, the lowest point, be taken as the origin. Let s denote the length of any arc OB; then, if p be the weight of a unit of length of the cord or chain, the load suspended, or the vertical tension, at B is sp. Let the horizontal tension be ap, the weight of a units of length of the chain. Let BD be a tangent at B; then, if BD represent the tension of the chain at B, BE and ED will represent respectively its horizontal and its vertical tension at B.

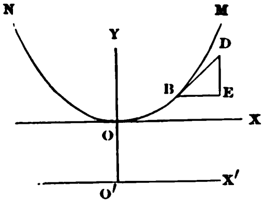

Fig. 25.

$$\text{Hence,}\quad \frac{dy}{dx} = \frac{\text{ED}}{\text{BE}} = \frac{sp}{ap} = \frac{s}{a}. \tag{1}$$

$$\therefore \frac{s}{a}\left[= \frac{dy}{dx}\right] = \frac{\sqrt{ds^2 - dx^2}}{dx}; \quad \therefore \frac{dx}{ds} = \frac{a}{\sqrt{a^2 + s^2}};$$

$$\therefore x = a\int \frac{ds}{\sqrt{a^2 + s^2}} = a \log(s + \sqrt{a^2 + s^2}) + C.$$

Since $x = 0$ when $s = 0$, $C = -a \log a$.

$$\therefore x = a \log\left(\frac{s}{a} + \sqrt{1 + \frac{s^2}{a^2}}\right). \tag{2}$$

Solving (2) for s, we have, for the length of the curve measured from O,

$$s = \frac{a}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right). \tag{3}$$

To find its equation, we have, from (1) and (3),

$$\frac{dy}{dx} = \tfrac{1}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right).$$

$$\therefore y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right) + C.$$

Since $y = 0$ when $x = 0$, $C = -a$.

$$\therefore y + a = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$$

is the equation of the catenary referred to OX and OY.

If O′O $= a$, and the curve be referred to the axes O′X′ and O′Y, its equation will evidently be

$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right).$$

CHAPTER VI.

INDETERMINATE FORMS.

77. When, for any particular value of its variable, a function assumes any one of the indeterminate forms,

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 0\cdot\infty,\ \infty-\infty,\ 0^0,\ \infty^0,\ 1^{\pm\infty},$$

the function, in the usual sense, has no value for this value of its variable. What we call the *value* of the function for this value of its variable is the *limit* which the function approaches, as the variable approaches this particular value as its limit.

Often, when a function assumes an indeterminate form, its value may be found by algebraic methods.

EXAMPLES.

1. Evaluate $\left.\frac{x^3-1}{x^2-1}\right]_1$; that is, find $\underset{x \doteq 1}{\text{limit}}\left[\frac{x^3-1}{x^2-1}\right]$.

In general,

$$\frac{x^3-1}{x^2-1}=\frac{x^2+x+1}{x+1};$$

$$\therefore \underset{x \doteq 1}{\text{limit}}\left[\frac{x^3-1}{x^2-1}\right]=\underset{x \doteq 1}{\text{limit}}\left[\frac{x^2+x+1}{x+1}\right]=\frac{3}{2}.$$

2. Evaluate $\left.\frac{(x-a)^{\frac{1}{3}}}{(x^2-a^2)^{\frac{1}{4}}}\right]_a$.

In general,

$$\frac{(x-a)^{\frac{1}{3}}}{(x^2-a^2)^{\frac{1}{4}}}=\frac{(x-a)^{\frac{4}{12}}}{(x-a)^{\frac{3}{12}}(x+a)^{\frac{1}{4}}}=\frac{(x-a)^{\frac{1}{12}}}{(x+a)^{\frac{1}{4}}};$$

$$\therefore \underset{x \doteq a}{\text{limit}}\left[\frac{(x-a)^{\frac{1}{3}}}{(x^2-a^2)^{\frac{1}{4}}}\right]=\underset{x \doteq a}{\text{limit}}\left[\frac{(x-a)^{\frac{1}{12}}}{(x+a)^{\frac{1}{4}}}\right]=0.$$

3. Evaluate $\left.\dfrac{(a^2-x^2)^{\frac{1}{2}}+(a-x)}{(a-x)^{\frac{1}{2}}+(a^3-x^3)^{\frac{1}{2}}}\right]_a$. *Ans.* $\dfrac{\sqrt{2a}}{1+a\sqrt{3}}$.

Evaluation by Differentiation.

78. *To evaluate* $\dfrac{\mathrm{f}(\mathrm{a})}{\phi(\mathrm{a})}$, or $\left.\dfrac{\mathrm{f}(\mathrm{x})}{\phi(\mathrm{x})}\right]_a$, *when it assumes the form* $\dfrac{0}{0}$.

$$\underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]=f'(x), \qquad \text{Ex. 1, p. 39.}$$

and
$$\underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\phi(x+\Delta x)-\phi(x)}{\Delta x}\right]=\phi'(x)\,;$$

$$\therefore \underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{f(x+\Delta x)-f(x)}{\phi(x+\Delta x)-\phi(x)}\right]=\frac{f'(x)}{\phi'(x)}. \qquad (1)$$

Substituting a for x in (1), and remembering that $f(a)$ and $\phi(a)$ are each 0, we have

$$\underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{f(a+\Delta x)}{\phi(a+\Delta x)}\right]=\frac{f'(a)}{\phi'(a)}, \text{ or } \frac{f(a)}{\phi(a)}=\frac{f'(a)}{\phi'(a)}.$$

If $f'(a)=0$, and $\phi'(a)$ is not 0, $\dfrac{f(a)}{\phi(a)}=0$.

If $f'(a)$ is not 0, and $\phi'(a)=0$, $\dfrac{f(a)}{\phi(a)}=\infty$.

If $f'(a)=0$, and $\phi'(a)=0$, $\dfrac{f'(a)}{\phi'(a)}$ also assumes the indeterminate form $\dfrac{0}{0}$. Applying to it the preceding process of reasoning, we have

$$\frac{f'(a)}{\phi'(a)}=\frac{f''(a)}{\phi''(a)}.$$

If this also assumes the form $\dfrac{0}{0}$, we pass to the next derivatives, and so on, until we obtain a fraction both of whose terms do not equal 0.

EXAMPLES.

Evaluate

1. $\left.\dfrac{\log x}{x-1}\right]_1$.

$$\left.\frac{\log x}{x-1}\right]_1=\frac{0}{0};\ \therefore \left.\frac{\log x}{x-1}\right]_1=\left.\frac{\frac{1}{x}}{1}\right]_1=1. \qquad \S\ 78.$$

2. $\left.\dfrac{1-\cos x}{x^2}\right]_0$.

$$\left.\frac{1-\cos x}{x^2}\right]_0=\frac{0}{0};\ \therefore \left.\frac{1-\cos x}{x^2}\right]_0=\left.\frac{\sin x}{2x}\right]_0=\left.\frac{\cos x}{2}\right]_0=\frac{1}{2}.$$

3. $\left.\dfrac{x-1}{x^n-1}\right]_1$. *Ans.* $\dfrac{1}{n}$.

4. $\left.\dfrac{e^x-e^{-x}}{\sin x}\right]_0$. 2.

5. $\left.\dfrac{e^x-e^{-x}-2x}{x-\sin x}\right]_0$. 2.

6. $\left.\dfrac{a^x-b^x}{x}\right]_0$. $\log\dfrac{a}{b}$.

7. $\left.\dfrac{x-\sin^{-1}x}{\sin^3 x}\right]_0$. $-\dfrac{1}{6}$.

79. *To evaluate* $\dfrac{f(a)}{\phi(a)}$, *when it assumes the form* $\dfrac{\infty}{\infty}$.

$$\underset{x\doteq a}{\text{limit}}\left[\frac{f(x)}{\phi(x)}\right]=\underset{x\doteq a}{\text{limit}}\left[\frac{\frac{1}{\phi(x)}}{\frac{1}{f(x)}}\right]=\underset{x\doteq a}{\text{limit}}\left[\frac{\frac{d}{dx}\left[\frac{1}{\phi(x)}\right]}{\frac{d}{dx}\left[\frac{1}{f(x)}\right]}\right]; \quad \S\ 78.$$

$$\therefore \underset{x\doteq a}{\text{limit}}\left[\frac{f(x)}{\phi(x)}\right]=\underset{x\doteq a}{\text{limit}}\left[\frac{[f(x)]^2}{[\phi(x)]^2}\cdot\frac{\phi'(x)}{f'(x)}\right]. \qquad (1)$$

Since, when the limits of two variables are equal, the limits of their equimultiples are equal; we have from (1), by multiplying by $\frac{\phi(x)}{f(x)}\frac{f'(x)}{\phi'(x)}$,

$$\underset{x \doteq a}{\text{limit}}\left[\frac{f(x)}{\phi(x)}\right] = \underset{x \doteq a}{\text{limit}}\left[\frac{f'(x)}{\phi'(x)}\right], \text{ or } \frac{f(a)}{\phi(a)} = \frac{f'(a)}{\phi'(a)}.$$

COR. From § 72, Cor., it follows that, if a function assumes the form $\frac{\infty}{\infty}$ for a finite value of the variable, all the functions obtained by the formula given above will also assume the form $\frac{\infty}{\infty}$. Hence, to evaluate the function, it is necessary to transform the original function, or some one of the derived functions, so that it will not assume the form $\frac{\infty}{\infty}$ for this finite value of the variable.

EXAMPLES.

Evaluate

1. $\left.\frac{\log x}{\operatorname{cosec} x}\right]_0$.

$$\left.\frac{\log x}{\operatorname{cosec} x}\right]_0 = \frac{-\infty}{\infty}; \quad \therefore \left.\frac{\log x}{\operatorname{cosec} x}\right]_0 = \left.\frac{\frac{1}{x}}{-\operatorname{cosec} x \cot x}\right]_0$$

$$= \left.\frac{-\sin^2 x}{x \cos x}\right]_0 = \left.\frac{-2 \sin x \cos x}{\cos x - x \sin x}\right]_0 = \frac{0}{1} = 0.$$

2. $\left.\frac{\log x}{x^a}\right]_\infty$. *Ans.* 0.

4. $\left.\frac{\log x}{x}\right]_\infty$. *Ans.* 0.

3. $\left.\frac{\cot x}{\log x}\right]_0$. ∞.

5. $\left.\frac{\tan x}{\tan 3x}\right]_{\frac{1}{2}\pi}$. 3.

80. The forms $0 \cdot \infty$ and $\infty - \infty$. Functions of x that assume the form $0 \cdot \infty$ or $\infty - \infty$ for a particular value of x, may be so transformed that they will assume the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ for the same value of x. Hence they can be evaluated by the previous methods.

EXAMPLES.

Evaluate

1. $\left.2^x \sin \frac{a}{2^x}\right]_\infty$.

Since $2^x \sin\frac{a}{2^x} = \dfrac{\sin\frac{a}{2^x}}{2^{-x}}$,

and $$\left.\frac{\sin\frac{a}{2^x}}{2^{-x}}\right]_\infty = \left. a\cos\frac{a}{2^x}\right]_\infty = a; \quad \therefore \left. 2^x \sin\frac{a}{2^x}\right]_\infty = a.$$

2. $\left.(1-x)\tan\frac{\pi x}{2}\right]_1$.

Since $(1-x)\tan\frac{\pi x}{2} = \dfrac{1-x}{\cot\frac{\pi x}{2}}$,

and $$\left.\frac{1-x}{\cot\frac{\pi x}{2}}\right]_1 = \left.\frac{1}{\frac{\pi}{2}\operatorname{cosec}^2\frac{\pi x}{2}}\right]_1 = \frac{2}{\pi}; \quad \therefore (1-x)\left.\tan\frac{\pi x}{2}\right]_1 = \frac{2}{\pi}.$$

3. $[\sec x - \tan x]_{\frac{1}{2}\pi}$.

Since $\sec x - \tan x = \dfrac{1}{\cos x} - \dfrac{\sin x}{\cos x} = \dfrac{1-\sin x}{\cos x}$,

and $$\left.\frac{1-\sin x}{\cos x}\right]_{\frac{1}{2}\pi} = \left.\frac{\cos x}{\sin x}\right]_{\frac{1}{2}\pi} = 0; \quad \therefore [\sec x - \tan x]_{\frac{1}{2}\pi} = 0.$$

4. $\left[\frac{2}{x^2-1} - \frac{1}{x-1}\right]_1$. *Ans.* $-\frac{1}{2}$.

5. $\left[\frac{1}{\log x} - \frac{x}{\log x}\right]_1$. -1.

81. The Forms 0^0, ∞^0, and $1^{\pm\infty}$. When, for any value of x, a function of x assumes one of the forms, 0^0, ∞^0, or $1^{\pm\infty}$, its logarithm assumes the form, $\pm 0\cdot\infty$, and the function is evaluated by evaluating its logarithm for this particular value of x.

EXAMPLES.

1. Evaluate $x^x\Big]_0$.

Since $\log x^x = x \log x = \dfrac{x}{\dfrac{1}{\log x}}$,

and $$\frac{x}{\dfrac{1}{\log x}}\Bigg]_0 = \frac{-(\log x)^2}{\dfrac{1}{x}}\Bigg]_0 = \frac{2\log x}{\dfrac{1}{x}}\Bigg]_0 = \frac{\dfrac{2}{x}}{-\dfrac{1}{x^2}}\Bigg]_0 = \frac{-2x}{1}\Bigg]_0 = 0;$$

$$\therefore \log x^x\Big]_0 = 0, \text{ or } x^x\Big]_0 = 1.$$

2. $x^{\sin x}\Big]_0$.

Since $\log x^{\sin x} = \sin x \log x = \dfrac{\log x}{\operatorname{cosec} x}$,

and $$\frac{\log x}{\operatorname{cosec} x}\Bigg]_0 = \frac{\dfrac{1}{x}}{-\operatorname{cosec} x \cot x}\Bigg]_0 = \frac{-\sin^2 x}{x\cos x}\Bigg]_0$$

$$= \frac{-2\sin x\cos x}{\cos x - x\sin x}\Bigg]_0 = 0;$$

$$\therefore \log x^{\sin x}\Big]_0 = 0, \text{ or } x^{\sin x}\Big]_0 = 1.$$

3. $\sin x^{\sin x}\Big]_0$. *Ans.* 1.

4. $\sin x^{\tan x}\Big]_{\frac{1}{2}\pi}$. *Ans.* 1.

82. Compound Indeterminate Forms. When the given function can be resolved into factors, some or each of which assumes the indeterminate form, each factor may be evaluated separately.

Thus, if the function be $\dfrac{(e^x-1)\tan^2 x}{x^3}\Bigg]_0$, we have

$$\left.\frac{(e^x-1)\tan^2 x}{x^3}\right]_0 = \left(\frac{\tan x}{x}\right)^2 \left.\frac{(e^x-1)}{x}\right]_0 = 1;$$

since $\left.\frac{\tan x}{x}\right]_0 = 1$, and $\left.\frac{e^x-1}{x}\right]_0 = 1.$

EXAMPLES.

1. $\left.\frac{\tan x - x}{x - \sin x}\right]_0 = 2.$

2. $\left.\frac{x - \sin x}{x^3}\right]_0 = \frac{1}{6}.$

3. $\left.\frac{e^x - e^{-x}}{\log(1+x)}\right]_0 = 2.$

4. $\left.\frac{e^x - e^{-x} - 2x}{(e^x-1)^3}\right]_0 = \frac{1}{3}.$

5. $\left.\frac{1 - \sin x + \cos x}{\sin x + \cos x - 1}\right]_{\frac{1}{2}\pi} = 1.$

6. $\left.\frac{x^2 + 2\cos x - 2}{x^4}\right]_0 = \frac{1}{12}.$

7. $\left.\frac{x^3 - 3x + 2}{x^4 - 6x^2 + 8x - 3}\right]_1 = \infty.$

8. $\left.\frac{a^{\sin x} - a}{\log \sin x}\right]_{\frac{1}{2}\pi} = a \log a.$

9. $\left. x^{\frac{1}{1-x}}\right]_1 = \frac{1}{e}.$

10. $\left.(\cos mx)^{\frac{n}{x}}\right]_0 = 1$

11. $\left. x \cdot e^{\frac{1}{x}}\right]_0 = \infty.$

12. $[x\tan x - \frac{1}{2}\pi \sec x]_{\frac{1}{2}\pi} = -1.$

13. $\left.\frac{\cos x\theta - \cos a\theta}{e^{-x^2\theta} - e^{-a^2\theta}}\right]_a = \frac{e^{a^2\theta}\sin a\theta}{2a}.$

14. $\left.\left(\frac{a}{x}+1\right)^x\right]_\infty = e^a.$

83. Evaluation of Derivatives of Implicit Functions. If an equation containing x and y is solved for y, y is an *explicit* function of x; if it is not so solved, y is an *implicit* function of x.

When y is an implicit function of x, its derivative, though containing both x and y, is a function of x. Hence, when the derivative assumes an indeterminate form for particular values of x and y, it can be evaluated by the previous methods.

EXAMPLES.

1. Find the slope of $a^2y^2 - a^2x^2 - x^4 = 0$ at $(0, 0)$.

Here $\dfrac{dy}{dx} = \dfrac{2a^2x + 4x^3}{2a^2y} = \dfrac{0}{0}$, when $x = y = 0$.

$$\text{Hence } \left.\frac{dy}{dx}\right]_{0,0} = \left.\frac{2a^2x + 4x^3}{2a^2y}\right]_{0,0} = \left.\frac{2a^2 + 12x^2}{2a^2\dfrac{dy}{dx}}\right]_{0,0} = \left.\frac{1}{\dfrac{dy}{dx}}\right]_{0,0};$$

$$\therefore \left.\left(\frac{dy}{dx}\right)^2\right]_{0,0} = 1, \text{ or } \left.\frac{dy}{dx}\right]_{0,0} = \pm 1.$$

2. Find the slope of $y^3 = ax^2 - x^3$ at $(0, 0)$.

$$\text{Here } \left.\frac{dy}{dx}\right]_{0,0} = \left.\frac{2ax - 3x^2}{3y^2}\right]_{0,0} = \left.\frac{2a - 6x}{6y\dfrac{dy}{dx}}\right]_{0,0};$$

$$\therefore \left.\left(\frac{dy}{dx}\right)^2\right]_{0,0} = \left.\frac{2a - 6x}{6y}\right]_{0,0} = \frac{2a}{0} = \infty, \text{ or } \left.\frac{dy}{dx}\right]_{0,0} = \pm \infty.$$

3. Find the slope of $x^3 - 3axy + y^3 = 0$ at $(0, 0)$.

Ans. $\left.\dfrac{dy}{dx}\right]_{0,0} = 0$ and ∞.

4. Find the slope of $x^4 - a^2xy + b^2y^2 = 0$ at $(0, 0)$.

Ans. $\left.\dfrac{dy}{dx}\right]_{0,0} = 0$ and $\dfrac{a^2}{b^2}$.

5. Find the slope of $(y^2 + x^2)^2 - 6axy^2 - 2ax^3 + a^2x^2 = 0$ at $(0, 0)$ and $(a, 0)$.

Ans. $\left.\dfrac{dy}{dx}\right]_{0,0} = \pm \infty$; $\left.\dfrac{dy}{dx}\right]_{a,0} = \pm \dfrac{1}{2}$.

CHAPTER VII.

DEVELOPMENT OF FUNCTIONS IN SERIES.

84. A **Series** is a succession of terms following one another according to some determinate law. The *sum* of a finite series is the sum of all its terms. An *infinite* series is one the number of whose terms is unlimited.

If the sum of the first n terms of an infinite series approaches a definite limit as n increases indefinitely, the series is **Convergent**; if not, it is **Divergent**.

The *limit* of the *sum* of the first n terms of an infinite convergent series, as n increases, is called the **Sum** of the series. An infinite divergent series has no definite sum.

85. To **Develop** a function is to find a series, the sum of which shall be equal to the function. Hence the *development* of a function is either a *finite* or an *infinite convergent* series.

For example, by division, we obtain

$$\frac{1-x^n}{1-x}=1+x+x^2+x^3+\cdots+x^{n-1}.$$

This finite series is evidently the development of $\frac{1-x^n}{1-x}$ for any value of x.

Again, by division, we obtain

$$\frac{1}{1-x}=1+x+x^2+x^3+\cdots+x^{n-1}+\frac{x^n}{1-x}.$$

Hence $\frac{x^n}{1-x}$ is the difference between $\frac{1}{1-x}$ and the sum of the first n terms of the series. For $x<+1$ and >-1, this difference evidently $\doteq 0$ as n increases, and the series is the development of the function. But for $x>+1$ or <-1, this difference increases numerically as n increases, and the series is not the development of the function. Thus, the series equals the function for $x=\frac{1}{2}$, but not for $x=+2$ or -2.

86. Taylor's Formula is a formula for developing a function of the sum of two variables in a series of terms arranged according to the ascending powers of one of the variables, with coefficients that are functions of the other.

A general symbol for any function of the sum of x and y is $f(x+y)$, of which $(x+y)^m$, $\log(x+y)$, a^{x+y}, $\sin(x+y)$, etc., are particular forms.

87. *To produce Taylor's Formula.*

We are to find the values of A, B, C, etc., when

$$f(x+y)=A+By+Cy^2+Dy^3+Ey^4+\cdots,$$

in which A, B, C, D, etc., are functions of x, but independent of y, the series being finite or convergent.

Let x' be any value of x, and A', B', C', etc., the corresponding values of A, B, C, etc.; then we have

$$f(x'+y)=A'+B'y+C'y^2+D'y^3+E'y^4+\cdots; \qquad (1)$$

$$\therefore\ f'(x'+y)=B'+2C'y+3D'y^2+4E'y^3+\cdots, \qquad (2)$$

$$f''(x'+y)=2C'+2\cdot 3D'y+3\cdot 4E'y^2+\cdots, \qquad (3)$$

$$f'''(x'+y)=2\cdot 3D'+2\cdot 3\cdot 4E'y+\cdots, \qquad (4)$$

$$f^{\text{IV}}(x'+y)=2\cdot 3\cdot 4E'+\cdots,\ \text{etc.} \qquad (5)$$

These equations, being true for all values of y, are true when $y=0$ (§ 6); hence we have

$$f(x')=A', \qquad f'(x')=B', \qquad f''(x')=2C',$$

$$f'''(x')=\underline{|3}\,D', \qquad f^{\text{IV}}(x')=\underline{|4}\,E', \qquad \text{etc.}$$

Solving these equations for A', B', C', etc., we have

$$A'=f(x'), \qquad B'=f'(x'), \qquad C'=\frac{f''(x')}{\underline{|2}},$$

$$D'=\frac{f'''(x')}{\underline{|3}}, \qquad E'=\frac{f^{\text{IV}}(x')}{\underline{|4}}, \qquad \text{etc.}$$

Substituting these values in (1), we obtain

$$f(x'+y)=f(x')+f'(x')y+f''(x')\frac{y^2}{\lfloor 2}+f'''(x')\frac{y^3}{\lfloor 3}$$
$$+f^{\text{iv}}(x')\frac{y^4}{\lfloor 4}+\cdots \qquad (6)$$

Since the coefficients in (6) are equal to $f(x)$, $f'(x)$, $f''(x)$, etc., for $x=x'$, and since x' is any value of x, we have, in general,

$$f(x+y)=f(x)+f'(x)y+f''(x)\frac{y^2}{\lfloor 2}+f'''(x)\frac{y^3}{\lfloor 3}$$
$$+f^{\text{iv}}(x)\frac{y^4}{\lfloor 4}+\cdots \qquad (\text{A})$$

This development of $f(x+y)$ was first published in 1715 by Dr. Brook Taylor, from whom it is named.

88. When $x'=0$, equation (6) of § 87 becomes

$$f(y)=f(0)+f'(0)y+f''(0)\frac{y^2}{\lfloor 2}+f'''(0)\frac{y^3}{\lfloor 3}$$
$$+f^{\text{iv}}(0)\frac{y^4}{\lfloor 4}+\cdots \qquad (7)$$

in which $f(0)$, $f'(0)$, etc., are the values of $f(y)$ and its successive derivatives when $y=0$.

Letting x represent the variable in (7), we obtain

$$f(x)=f(0)+f'(0)x+f''(0)\frac{x^2}{\lfloor 2}+f'''(0)\frac{x^3}{\lfloor 3}$$
$$+f^{\text{iv}}(0)\frac{x^4}{\lfloor 4}+\cdots \qquad (\text{B})$$

Equation (B), called **Maclaurin's Formula**, is a formula for developing a function of a single variable in a series of terms arranged according to the ascending powers of that variable, with constant coefficients.

The completion* of Taylor's and Maclaurin's formulas will be deferred until we have applied them to the development of a few functions.

89. *To develop* $(x+y)^m$, *or to deduce the Binomial Theorem.*

Here $f(x+y) = (x+y)^m$; $\therefore f(x) = x^m$,

$$f'(x) = mx^{m-1}, \qquad f''(x) = m(m-1)x^{m-2},$$

$$f'''(x) = m(m-1)(m-2)x^{m-3}, \text{ etc.}$$

Substituting these values in Taylor's formula, we have

$$(x+y)^m = x^m + mx^{m-1}y + \frac{m(m-1)}{\underline{|2}}x^{m-2}y^2 + \frac{m(m-1)(m-2)}{\underline{|3}}x^{m-3}y^3 + \cdots$$

90. *To develop* $\log_a(x+y)$.

Here $f(x+y) = \log_a(x+y)$; $\therefore f(x) = \log_a x$,

$$f'(x) = \frac{m}{x}, \quad f''(x) = -\frac{m}{x^2}, \quad f'''(x) = \frac{2m}{x^3}, \text{ etc.}$$

Substituting these values in Taylor's formula, we have

$$\log_a(x+y) = \log_a x + m\left(\frac{y}{x} - \frac{y^2}{2x^2} + \frac{y^3}{3x^3} - \frac{y^4}{4x^4} + \cdots\right),$$

which is the *logarithmic series.*

Cor. If $x = 1$, and $m = 1$, we have

$$\log(1+y) = \frac{y}{1} - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots, \quad \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9}$$

which is the *Naperian logarithmic series.*

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* The series obtained by applying Taylor's or Maclaurin's formula, as given above, to any given function may or may not be the development of that function. Their *complete* forms, however, enable us to determine what functions can be developed by them.

91. *To develop* a^{x+y}.

Here $f(x) = a^x$, $f'(x) = a^x \log a$, etc.;

$$\therefore a^{x+y} = a^x\left[1 + \log a\frac{y}{1} + (\log a)^2\frac{y^2}{\underline{|2}} + (\log a)^3\frac{y^3}{\underline{|3}} + \cdots\right]$$

92. *To develop* $(a + x)^m$ *by Maclaurin's Formula.*

$$\begin{array}{lll} \text{Here} & f(x) = (a+x)^m; & \therefore\ f(0) = a^m. \\ & f'(x) = m(a+x)^{m-1}; & \therefore\ f'(0) = ma^{m-1}. \\ & f''(x) = m(m-1)(a+x)^{m-2}; & \therefore f''(0) = m(m-1)a^{m-2}. \\ & \quad\text{etc.} & \quad\text{etc.} \end{array}$$

Substituting these values in Maclaurin's formula, we have

$$(a + x)^m = a^m + ma^{m-1}x + \frac{m(m-1)}{\underline{|2}}a^{m-2}x^2 + \cdots$$

93. *To develop* $\sin x$.

$$\begin{array}{lll} \text{Here} & f(x) = \sin x; & \therefore\ f(0) = 0. \\ & f'(x) = \cos x; & \therefore\ f'(0) = 1. \\ & f''(x) = -\sin x; & \therefore\ f''(0) = 0. \\ & f'''(x) = -\cos x; & \therefore f'''(0) = -1. \\ & f^{\text{iv}}(x) = \sin x; & \therefore f^{\text{iv}}(0) = 0. \\ & f^{\text{v}}(x) = \cos x; & \therefore\ f^{\text{v}}(0) = 1. \\ & \quad\text{etc.} & \quad\text{etc.} \end{array}$$

Substituting these values in the formula, we have

$$\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \frac{x^9}{\underline{|9}} - \cdots$$

94. *To develop* $\cos x$.

$$\cos x = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} + \frac{x^8}{\underline{|8}} - \cdots$$

This result could be obtained by differentiating the value of $\sin x$ found in § 93.

95. *To develop* a^x.

$$a^x = 1 + \log a \frac{x}{1} + (\log a)^2 \frac{x^2}{\underline{|2}} + (\log a)^3 \frac{x^3}{\underline{|3}} + \cdots,$$

which is the *exponential series.*

Cor. 1. If $a = e$, the Naperian base, we have

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \cdots$$

Cor. 2. Putting $x = 1$, we have

$$e = 1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \frac{1}{\underline{|5}} + \cdots$$

Hence $e = 2.718281 +$.

96. *To develop* $\log_a(1+x)$.

$$\log_a(1+x) = m\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right), \qquad (1)$$

which is the *common logarithmic series.*

If $m = 1$, we have

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots, \qquad (1')$$

which is the *Naperian logarithmic series.*

In § 104, this development is proved to hold only for values of x between -1 and $+1$; hence, in this form, it is useless for the computation of Naperian logarithms of numbers greater than 2. We therefore proceed to adapt the Naperian logarithmic series to the computation of logarithms.

Substituting $-x$ for x in (1′), we have

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots \qquad (2)$$

Subtracting (2) from (1′), we have

$$\log(1+x) - \log(1-x) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right). \qquad (3)$$

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Let $x = \frac{1}{2z+1}$; then $\frac{1+x}{1-x} = \frac{z+1}{z}$; and, for any positive value of z, $x < 1$.

Hence $\log(1+x) - \log(1-x) = \log(z+1) - \log z$.

Substituting these values in (3), we have

$$\log(z+1) - \log z$$
$$= 2\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \cdots\right). \quad (4)$$

Equation (4) is true for any positive value of z; and, since the series converges rapidly, $\log(z+1)$ can be readily computed when $\log z$ is known.

Putting $z = 1$ in (4), we obtain

$$\log 2 = 2\left(\frac{1}{3} + \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} + \frac{1}{7 \cdot 3^7} + \cdots\right).$$

Summing six terms of this series, we find

$$\log 2 = 0.693147+.$$

Putting $z = 2$ in (4), we have

$$\log 3 = \log 2 + 2\left(\frac{1}{5} + \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} + \frac{1}{7 \cdot 5^7} + \cdots\right)$$
$$= 1.098612+.$$
$$\log 4 = 2 \log 2 = 1.386294+.$$

Putting $z = 4$, we obtain

$$\log 5 = \log 4 + 2\left(\frac{1}{9} + \frac{1}{3 \cdot 9^3} + \frac{1}{5 \cdot 9^5} + \frac{1}{7 \cdot 9^7} + \cdots\right)$$
$$= 1.6094379+.$$
$$\log 10 = \log 5 + \log 2 = 2.302585+.$$

In this way we can compute the Naperian logarithms of all numbers.

COR. 1. Letting m and m' be the moduli of two systems of logarithms whose bases are a and a', from (1) we evidently have

$$\frac{\log_a(1+x)}{\log_{a'}(1+x)} = \frac{m}{m'}, \quad (5)$$

in which x lies between -1 and $+1$.

To prove that the principle in (5) is true for all numbers, let

$$\log_a(1+x)=u, \text{ and } \log_{a'}(1+x)=w;$$

then $$a^u=(1+x)=a'^w, \text{ or } a'=a^{\frac{u}{w}}; \tag{6}$$

and $$\frac{\log_a(1+x)}{\log_{a'}(1+x)}=\frac{u}{w}. \tag{7}$$

Again, y being any number, let

$$\log_a y=z, \text{ and } \log_{a'}y=v;$$

then $$a^z=y=a'^v, \text{ or } a'=a^{\frac{z}{v}}; \tag{8}$$

and $$\frac{\log_a y}{\log_{a'}y}=\frac{z}{v}. \tag{9}$$

From (6) and (8),

$$a^{\frac{u}{w}}=a^{\frac{z}{v}}, \text{ or } \frac{u}{w}=\frac{z}{v}. \tag{10}$$

From (7), (9), (10), and (5), we have

$$\frac{\log_a y}{\log_{a'}y}=\frac{\log_a(1+x)}{\log_{a'}(1+x)}=\frac{m}{m'}. \tag{11}$$

Hence, *the logarithms of the same number in different systems are proportional to the moduli of those systems.*

Cor. 2. If, in (11) of Cor. 1, we let $a'=e$, we have

$$m'=1,$$

and $$\log_a y=m\log y. \tag{12}$$

Hence, *the logarithm of a number in any system is equal to the Naperian logarithm of the same number multiplied by the modulus of that system.*

Cor. 3. If, in (12) of Cor. 2, $y=a$, we have

$$m=\frac{1}{\log a}.$$

Hence, *the modulus of any system of logarithms is the reciprocal of the Naperian logarithm of its base.*

In the Common system, $a = 10$;

hence $$m = \frac{1}{\log 10} = \frac{1}{2.302585} = .434294+.$$

97. Taylor's formula evidently fails to develop $f(x+y)$ for $x=a$, if $f(a)$ or $f^n(a)$ is infinite, while n remains finite; while Maclaurin's fails to develop $f(x)$ for any value of x, if $f(0)$ or $f^n(0)$ is infinite for n finite.

For example, by Taylor's formula, we have

$$(x-b+y)^{\frac{1}{2}} = (x-b)^{\frac{1}{2}} + \frac{y}{2(x-b)^{\frac{1}{2}}} - \frac{y^2}{8(x-b)^{\frac{3}{2}}} + \cdots \quad (1)$$

When $x=b$, (1) becomes

$$\sqrt{y} = \infty - \infty + \cdots$$

Hence the formula fails to develop $(x-b+y)^{\frac{1}{2}}$ for $x=b$.

By Maclaurin's formula, we have

$$\log x = -\infty + \infty - \infty + \cdots$$

Hence Maclaurin's formula fails to develop $\log x$ for any value of x.

98. To complete Taylor's and Maclaurin's formulas so that they shall enable us to determine what functions can be developed by them, we need the following lemma:

Lemma. *If* f(x) *is continuous between* x = a *and* x = b, *and if* f(a) = f(b) = 0, *then* f'(x), *if continuous, must equal zero for some value of* x *between* a *and* b.

For, if $f(x)$ is continuous, and $f(a)=f(b)=0$; then, as x changes from a to b, $f(x)$ must first increase, and then decrease; or first decrease, and then increase; hence $f'(x)$ must change from + to −, or from − to +, and therefore, if continuous, pass through 0.

99. Completion of Taylor's and Maclaurin's Formulas. To be the development of $f(x+y)$, the series in Taylor's formula must be finite or infinite and convergent (§ 85).

Let $P\frac{y^n}{\underline{|n}}$ be the *difference* between $f(x+y)$ and the sum of the first n terms of the series; then we have

$$f(x+y)=f(x)+f'(x)\frac{y}{1}+f''(x)\frac{y^2}{\underline{|2}}+f'''(x)\frac{y^3}{\underline{|3}}$$
$$+\cdots+f^{n-1}(x)\frac{y^{n-1}}{\underline{|n-1}}+P\frac{y^n}{\underline{|n}}. \qquad (1)$$

We proceed to find the value of P.

Letting $y=X-x$ in (1), and transposing, we have

$$f(X)-f(x)-f'(x)\frac{X-x}{1}-f''(x)\frac{(X-x)^2}{\underline{|2}}$$
$$-f'''(x)\frac{(X-x)^3}{\underline{|3}}-\cdots-f^{n-1}(x)\frac{(X-x)^{n-1}}{\underline{|n-1}}$$
$$-P\frac{(X-x)^n}{\underline{|n}}=0. \qquad (2)$$

Let $F(z)$ represent the function of z obtained by substituting z for x in the first member of (2); then

$$F(z)=f(X)-f(z)-f'(z)\frac{X-z}{1}-f''(z)\frac{(X-z)^2}{\underline{|2}}$$
$$-\cdots-f^{n-1}(z)\frac{(X-z)^{n-1}}{\underline{|n-1}}-P\frac{(X-z)^n}{\underline{|n}}. \qquad (3)$$

Substituting X for z in (3), we obtain $F(X)=0$.

From (2) we see that the right-hand member of (3) is 0 for $z=x$; hence, by substituting x for z in (3), we obtain $F(x)=0$.

Differentiating (3) to obtain $F'(z)$, we find that the terms of the second member destroy each other in pairs, with the exception of the last two, and obtain

$$F'(z) = -\frac{(X-z)^{n-1}}{\underline{|n-1}} f^n(z) + \frac{(X-z)^{n-1}}{\underline{|n-1}} P.$$

Whence, $P = f^n(z)$ when $F'(z) = 0$. Since $F(z) = 0$ when $z = X$, and also when $z = x$, $F'(z) = 0$ for some value of z between X and x (§ 98). Now, by giving to θ some value between 0 and $+1$, any value between x and X can evidently be represented by $x + \theta(X - x)$.

Hence, $P = f^n[x + \theta(X - x)] = f^n(x + \theta y)$.

Substituting this value of P in (1), we have

$$f(x+y) = f(x) + f'(x)y + f''(x)\frac{y^2}{\underline{|2}} + f'''(x)\frac{y^3}{\underline{|3}} + \cdots$$
$$+ f^{n-1}(x)\frac{y^{n-1}}{\underline{|n-1}} + f^n(x+\theta y)\frac{y^n}{\underline{|n}},$$

which is one complete form of Taylor's formula.

Cor. Letting $x = 0$, and putting x for y, we have

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{\underline{|2}} + f'''(0)\frac{x^3}{\underline{|3}} + \cdots$$
$$+ f^{n-1}(0)\frac{x^{n-1}}{\underline{|n-1}} + f^n(\theta x)\frac{x^n}{\underline{|n}},$$

which is one complete form of Maclaurin's formula.

100. If we had let $P_1 y$ be the difference between $f(x+y)$ and the sum of the first n terms of the series in Taylor's formula, we should have found

$$P_1 = f^n(x+\theta y)\frac{y^{n-1}(1-\theta)^{n-1}}{\underline{|n-1}}.$$

Hence a second complete form of Taylor's formula is

$$f(x+y) = f(x) + f'(x)\frac{y}{1} + f''(x)\frac{y^2}{\underline{|2}} + \cdots$$
$$+ f^{n-1}(x)\frac{y^{n-1}}{\underline{|n-1}} + f^n(x+\theta y)\frac{(1-\theta)^{n-1}y^n}{\underline{|n-1}}.$$

COR. 1. The corresponding complete form of Maclaurin's formula is evidently

$$f(x)=f(0)+f'(0)x+f''(0)\frac{x^2}{\lfloor 2}+\cdots+f^{n-1}(0)\frac{x^{n-1}}{\lfloor n-1}$$
$$+f^n(\theta x)\frac{(1-\theta)^{n-1}x^n}{\lfloor n-1}.$$

All that we know of θ in any of these formulas is that its value lies between 0 and +1.

COR. 2. If, on applying to a given function any one of these completed formulas, the last term becomes 0, or approaches 0 as its limit, as n increases, the formula evidently develops this function; if not, the formula fails.

101. Since $\frac{y^n}{\lfloor n}=\frac{y}{n}\cdot\frac{y^{n-1}}{\lfloor n-1}$; and since $\frac{y}{n}$ is very small when y is finite, and n is very large; each value of $\frac{y^n}{\lfloor n}$ is a very small part of its preceding value.

Hence $\frac{y^n}{\lfloor n}\doteq 0$, *when* y *is finite and* n *increases indefinitely.*

COR. *If* $f^n(x)$ *does not become infinite with* n, *Taylor's and Maclaurin's formulas give the true development of* $f(x+y)$ *and* $f(x)$ *respectively.*

102. *To prove that Maclaurin's formula develops* a^x.

Here $f^n(x)=(\log a)^n a^x$; $\therefore f^n(\theta x)=(\log a)^n a^{\theta x}$,

and $f^n(\theta x)\frac{x^n}{\lfloor n}=\frac{(x\log a)^n}{\lfloor n}a^{\theta x}.$

Since $a^{\theta x}$ is finite, and $\frac{(x\log a)^n}{\lfloor n}\doteq 0$ as n increases indefinitely (§ 101);

$$\frac{(x\log a)^n}{\lfloor n}a^{\theta x}\doteq 0$$

as n increases, and the formula develops the function (§ 100, Cor. 2).

103. *To prove that Maclaurin's formula develops sin* x *and cos* x.

The nth derivative of each of these functions is finite, however great n may be; hence Maclaurin's formula develops both of them (§ 101, Cor.).

104. *To determine for what values of* x *Maclaurin's formula develops log*(1 + x).

The formula gives (§ 96),

$$\log(1+x)=\frac{x}{1}-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots+\frac{(-1)^{n-2}x^{n-1}}{n-1}$$
$$+\frac{(-1)^{n-1}x^n}{n}.$$

The ratio of the nth term to the term before it evidently approaches $-x$ as n increases. Hence, if x is numerically greater than 1, the series is divergent, and cannot be the development of $\log(1+x)$. We need, therefore, to examine the value of the last term of the formula only for values of x between -1 and $+1$.

$$f^n(x)=\frac{(-1)^{n-1}\lfloor\underline{n-1}}{(1+x)^n};$$

$$\therefore f^n(\theta x)\frac{x^n}{\lfloor\underline{n}}=\frac{(-1)^{n-1}}{n}\left(\frac{x}{1+\theta x}\right)^n.$$

For values of x between 0 and $+1$, $\frac{(-1)^{n-1}}{n}$ and $\left(\frac{x}{1+\theta x}\right)^n$ each approaches 0 as n increases; hence the formula develops $\log(1+x)$ for these values of x.

When x lies between 0 and -1, let x_1 represent its absolute value; then $x=-x_1$, and $\log(1+x)=\log(1-x_1)$.

Using the second form of the formula, we have, numerically,

$$f^n(\theta x_1)\frac{(1-\theta)^{n-1}x_1^n}{\lfloor\underline{n-1}}=\frac{(1-\theta)^{n-1}x_1^n}{(1-\theta x_1)^n}$$
$$=\left(\frac{x_1-\theta x_1}{1-\theta x_1}\right)^{n-1}\frac{x_1}{1-\theta x_1}.$$

For values of x_1 between 0 and $+1$, $\frac{x_1}{1-\theta x_1}$ is finite, and $\left(\frac{x_1-\theta x_1}{1-\theta x_1}\right)^{n-1}$ approaches 0 as n increases. The formula therefore develops $\log(1+x)$ when x lies between 0 and -1.

Hence, *Maclaurin's formula develops* $\log(1+\mathrm{x})$ *for values of* x *between* -1 *and* $+1$.

105. *To determine for what values of* x *Maclaurin's formula develops* $(1+\mathrm{x})^{\mathrm{m}}$.

The formula gives

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{\underline{|2}}x^2 + \cdots$$
$$+ \frac{m(m-1)\cdots(m-n+2)}{\underline{|n-1}}x^{n-1}.$$

If m is a positive whole number, $f^n(\theta x)\frac{x^n}{\underline{|n}} = 0$ when $n = m+1$; hence, in this case, Maclaurin's formula develops $(1+x)^m$ in a finite series of $m+1$ terms.

If m is negative or fractional, the series is infinite. The ratio of the $(n+1)$th term to the nth is $\frac{m-n+1}{n}x$, which approaches $-x$ as n increases. The series therefore is divergent, and cannot equal $(1+x)^m$ when x is numerically greater than 1. Hence we need examine the value of the last term of the formula only for values of x between $+1$ and -1.

Here $\quad f^n(x) = \frac{m(m-1)\cdots(m-n+1)(1+x)^m}{(1+x)^n};$

$$\therefore f^n(\theta x)\frac{x^n}{\underline{|n}}$$
$$= \left[\frac{m(m-1)\cdots(m-n+1)}{\underline{|n}}x^n\right]\left(\frac{1}{1+\theta x}\right)^n(1+\theta x)^m.$$

When x lies between 0 and 1, $(1+\theta x)^m$ is finite,

and $\quad \left(\frac{1}{1+\theta x}\right)^n \doteq 0$, as n increases indefinitely.

An increase of 1 in the number of terms multiplies the factor in brackets by $\frac{m-n}{n+1}x$, which approaches $-x$ as n increases. Hence the last term of the formula approaches 0 as n increases; and the formula develops $(1+x)^m$ for values of x between 0 and $+1$.

Using the second form of the formula, we have

$$f^n(\theta x)\frac{(1-\theta)^{n-1}x^n}{\underline{|n-1}}$$

$$=\left[\frac{m(m-1)\cdots(m-n+1)}{\underline{|n-1}}x^n\right]\left(\frac{1-\theta}{1+\theta x}\right)^n\frac{(1+\theta x)^m}{1-\theta}.$$

For values of x between 0 and -1, $\frac{(1+\theta x)^m}{1-\theta}$ is finite; $\left(\frac{1-\theta}{1+\theta x}\right)^n$ approaches 0 as n increases; and an increase of 1 in the number of terms multiplies the factor in brackets by $\frac{m-n}{n}x$, which approaches $-x$ as n increases. Hence Maclaurin's formula develops $(1+x)^m$ for all values of x between -1 and $+1$.

106. The Binomial Theorem. Since $(a+x)^m = a^m\left(1+\frac{x}{a}\right)^m$, and $\left(1+\frac{x}{a}\right)^m$ can be developed by Maclaurin's formula, when x is numerically less than a (§ 105); therefore, in this case,

$$(a+x)^m = a^m + ma^{m-1}x + \frac{m(m-1)}{\underline{|2}}a^{m-2}x^2 + \cdots \quad (1)$$

For like reason, when x is numerically greater than a,

$$(x+a)^m = x^m + mx^{m-1}a + \frac{m(m-1)}{\underline{|2}}x^{m-2}a^2 + \cdots \quad (2)$$

Hence one, and only one, form of the development of $(a+x)^m$ holds for any set of values of a and x.

107. *To develop* $\tan^{-1}\mathrm{x}$, *and find the value of* π.

$$\tan^{-1}x = \int \frac{dx}{1+x^2}.$$

When x lies between -1 and $+1$,

$$\frac{1}{1+x^2} = (1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots; \ \S\ 105.$$

$$\therefore \int \frac{dx}{1+x^2} = \int dx - \int x^2 dx + \int x^4 dx - \int x^6 dx + \int x^8 dx - \cdots$$

Hence, if x is numerically less than 1,

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots \qquad (1)$$

$C = 0$, since $\tan^{-1}x = 0$ when $x = 0$.

If we put $x = \sqrt{\frac{1}{3}}$, equation (1) becomes

$$\tan^{-1}\sqrt{\frac{1}{3}} = \frac{\pi}{6} = \sqrt{\frac{1}{3}}\left(1 - \frac{1}{9} + \frac{1}{45} - \frac{1}{189} + \cdots\right);$$

$$\therefore \pi = 2\sqrt{3}\left(1 - \frac{1}{9} + \frac{1}{45} - \frac{1}{189} + \cdots\right) = 3.141592+.$$

108. *To develop* $\sin^{-1}\mathrm{x}$, *and find the value of* π.

$$\sin^{-1}x = x + \frac{x^3}{2\cdot 3} + \frac{1\cdot 3x^5}{2\cdot 4\cdot 5} + \frac{1\cdot 3\cdot 5x^7}{2\cdot 4\cdot 6\cdot 7} + \cdots,$$

when x lies between -1 and $+1$.

$$\therefore \frac{\pi}{6} = \frac{1}{2}\left(1 + \frac{1}{24} + \frac{3}{640} + \frac{5}{7168} + \cdots\right);$$

or, $\pi = 3.141592+.$

It was by means of this series that Sir Isaac Newton computed the value of π.

109. *To prove geometrically that* $\mathrm{f}(\mathrm{a}+\mathrm{h})=\mathrm{f}(\mathrm{a})+\mathrm{h}\,\mathrm{f}'(\mathrm{a}+\theta\mathrm{h})$, $\mathrm{f}(\mathrm{x})$ *and* $\mathrm{f}'(\mathrm{x})$ *being continuous and finite between* $\mathrm{x}=\mathrm{a}$ *and* $\mathrm{x}=\mathrm{a}+\mathrm{h}$.

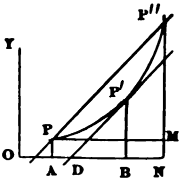

Fig. 26.

Let PP′P″ be the locus of $y=f(x)$, $a=$ OA, and $h=$ AN; then $f(a)=$ AP $=$ NM, and $f(a+h)=$ NP″. The curve must be parallel to the chord PP″ at some point P′ between P and P″. Now, OB $=$ OA $+$ AB $=a+\theta h$, θ having some value between 0 and 1;

hence $$f'(a+\theta h)=\tan \mathrm{BDP}'=\tan \mathrm{MPP}''.$$

$$\therefore \mathrm{MP}''=\mathrm{PM}\tan\mathrm{MPP}''=hf'(a+\theta h);$$

$$\therefore f(a+h)=\mathrm{NM}+\mathrm{MP}''=f(a)+hf'(a+\theta h).$$

EXAMPLES.

1. Develop $(a^2+bx^2)^{\frac{1}{2}}$.

Ans. $$(a^2+bx^2)^{\frac{1}{2}}=a+\frac{bx^2}{2a}-\frac{b^2x^4}{8a^3}+\frac{b^3x^6}{16a^5}-\cdots$$

2. Prove that $\tan x=x+\dfrac{x^3}{3}+\dfrac{2x^5}{15}+\cdots$

3. Prove that $\sec x=1+\dfrac{x^2}{2}+\dfrac{5x^4}{24}+\dfrac{61x^6}{720}+\cdots$

4. Given $f(x)=2x^3-3x^2+4x-3$, to find the value of $f(x+h)$, h being a variable increment of x.

Here $f(x)$ of Taylor's formula is

$$2x^3-3x^2+4x-3, \text{ and } y=h;$$

$$\therefore\ f'(x)=6x^2-6x+4,\quad f''(x)=12x-6,$$

$$f'''(x)=12,\qquad \text{and } f^{\text{iv}}(x)=0.$$

$$\therefore f(x+h)=2x^3-3x^2+4x-3+(6x^2-6x+4)h$$
$$+(6x-3)h^2+2h^3.$$

5. Given $f(x)=2x^5-3x$, to find $f(x+h)$.

6. Develop $\sin(x+y)$.

$$\begin{aligned}\sin(x+y) &= \sin x\left(1-\frac{y^2}{\underline{|2}}+\frac{y^4}{\underline{|4}}-\frac{y^6}{\underline{|6}}+\cdots\right)\\ &\quad+\cos x\left(y-\frac{y^3}{\underline{|3}}+\frac{y^5}{\underline{|5}}-\frac{y^7}{\underline{|7}}+\cdots\right)\\ &= \sin x\cos y+\cos x\sin y. \qquad \S\S\ 93,\ 94.\end{aligned}$$

7. Prove that $\cos(x+y)=\cos x\cos y-\sin x\sin y$.

8. Prove that $e^{\cos x}=e\left(1-\frac{x^2}{\underline{|2}}+\frac{4x^4}{\underline{|4}}-\frac{31x^6}{\underline{|6}}+\cdots\right)$.

9. Prove that $\log(1+\sin x)=x-\frac{x^2}{2}+\frac{x^3}{6}-\frac{x^4}{12}+\cdots$

10. Develop $e^{\sin x}$. $\quad e^{\sin x}=1+x+\frac{x^2}{\underline{|2}}-\frac{3x^4}{\underline{|4}}+\frac{8x^5}{\underline{|5}}-\frac{3x^6}{\underline{|6}}+\cdots$

11. Develop $\frac{a}{\sqrt{b^2-c^2x^2}}$.

12. Develop x^2e^x. $\quad x^2e^x=x^2+x^3+\frac{x^4}{\underline{|2}}+\frac{x^5}{\underline{|3}}+\cdots$

13. Develop $\log(1+e^x)$. $\quad \log 2+\frac{x}{2}+\frac{x^2}{2^3}-\frac{x^4}{2^3\underline{|4}}+\cdots$

14. Prove that, if $f(x)=f(-x)$, the development of $f(x)$ contains only even powers of x; while, if $f(x)=-f(-x)$, the development of $f(x)$ involves only odd powers of x.

15. What powers of x appear in the development of $\sin x$, and why? $\cos x$? $\tan x$? $\sec x$? $\sin^{-1}x$? $\tan^{-1}x$?

CHAPTER VIII.

MAXIMA AND MINIMA.

110. A **maximum** value of a function of a single variable is a value that is *greater* than its immediately preceding and succeeding values. A **minimum** value is one that is *less* than its immediately preceding and succeeding values.

Therefore, if we conceive x as always increasing, $f(x)$ must be an *increasing* function immediately *before*, and a *decreasing* function immediately *after*, a maximum; also, immediately before a minimum, $f(x)$ is a *decreasing*, and immediately after an *increasing* function.

Hence, $f'(x)$ *is positive before and negative after a maximum of* $f(x)$, *and negative before and positive after a minimum.*

111. From § 110, $f'(x)$ must change its sign as $f(x)$ passes through either a maximum or a minimum. But, to change its sign, $f'(x)$ must pass through 0 or ∞.

Hence, *any value of* x *that renders* $f(x)$ *a maximum or a minimum is a root of* $f'(x)=0$ *or* $f'(x)=\infty$.

The converse of this theorem is not true; that is, any root of $f'(x)=0$ or ∞ does not necessarily render $f(x)$ a maximum or a minimum. These roots are simply the *critical* values of x, for each of which the function is to be examined.

To illustrate geometrically the preceding definitions and principles, suppose $a'h'$ to be the locus of $y=f(x)$. Then, by definition, aa', cc', and ee' are maxima, and bb' and dd' are minima of $f(x)$. In passing along the curve from left to right, the slope of the curve $f'(x)$ is positive before, and negative after, a maximum ordinate; and negative before, and positive after, a minimum ordinate.

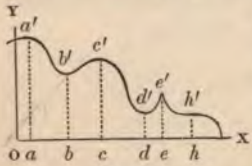

Fig. 27.

Moreover, at a point whose ordinate $f(x)$ is a maximum or a minimum, the curve is either parallel or perpendicular to the axis of x, and therefore $f'(x)=0$ or ∞.

At $x=oh$, $f'(x)=0$, but hh' is neither a maximum nor a minimum of $f(x)$.

112. Whether any one of the *critical* values of x renders $f(x)$ a maximum or a minimum can be determined by one of the following methods:

First Method. In this method we determine *directly* whether $f'(x)$ changes from positive to negative, or from negative to positive, as x passes through a *critical* value.

Let a be a *critical* value, and h a very small quantity. In $f'(x)$ substitute $a-h$ and $a+h$ for x.

If $f'(a-h)$ is positive, and $f'(a+h)$ negative,
$f(a)$ is a maximum. § 110

If $f'(a-h)$ is negative, and $f'(a+h)$ positive,
$f(a)$ is a minimum.

If $f'(a-h)$ and $f'(a+h)$ have the same sign,
$f(a)$ is neither a maximum nor a minimum.

Second Method. This method applies only to the roots of $f'(x)=0$. Let a be a *critical* value of x. Developing $f(x-h)$ and $f(x+h)$ by Taylor's formula, substituting a for x, transposing $f(a)$, and remembering that $f'(a)=0$, we have

$$f(a-h)-f(a)=f''(a)\frac{h^2}{\underline{|2}}-f'''(a)\frac{h^3}{\underline{|3}}+f^{\text{IV}}(a)\frac{h^4}{\underline{|4}}-\cdots, \quad (1)$$

and $$f(a+h)-f(a)=f''(a)\frac{h^2}{\underline{|2}}+f'''(a)\frac{h^3}{\underline{|3}}+f^{\text{IV}}(a)\frac{h^4}{\underline{|4}}+\cdots \quad (2)$$

If h be taken very small, the sign of the second member of either (1) or (2) will be the same as the sign of its first term. Hence, if $f''(a)$ is negative, $f(a)$ is greater than both $f(a-h)$ and $f(a+h)$, and therefore a maximum; while, if $f''(a)$ is positive, $f(a)$ is less than both $f(a-h)$ and $f(a+h)$, and therefore a minimum. If $f''(a)$ is 0, and $f'''(a)$ is not 0, $f(a)$ is neither greater than both $f(a-h)$ and $f(a+h)$, nor less than both, and is therefore neither a maximum nor a minimum. If $f'''(a)$, as well as $f''(a)$, is 0, and $f^{\text{IV}}(a)$ is negative, $f(a)$ is greater than both $f(a-h)$ and $f(a+h)$, and therefore a maximum; while, if $f^{\text{IV}}(a)$ is positive, $f(a)$ is a minimum, and so on.

Hence, *if* a *is a critical value obtained from* $f'(x) = 0$, *substitute* a *for* x *in the successive derivatives of* $f(x)$. *If the first derivative that does not reduce to* 0 *is of an odd order,* $f(a)$ *is neither a maximum nor a minimum; but, if the first derivative that does not reduce to* 0 *is of an even order,* $f(a)$ *is a maximum or a minimum, according as this derivative is negative or positive.*

113. *Maxima and minima occur alternately.*

Suppose that $a < b$, and that $f(a)$ and $f(b)$ are maxima of $f(x)$. When $x = a + h$, $f(x)$ is decreasing; and, when $x = b - h$, $f(x)$ is increasing, h being very small. But, in passing from a decreasing to an increasing state, $f(x)$ must pass through a minimum. Hence, between two maxima, there must be at least one minimum.

In like manner, it can be proved that between two minima there must be at least one maximum.

114. The solution of problems in maxima and minima is sometimes facilitated by the following considerations:

(*a*) Any value of x that renders $c \cdot f(x)$ a maximum or a minimum, c being positive, renders $f(x)$ a maximum or a minimum.

(*b*) Any value of x that renders $\log_a f(x)$ a maximum or a minimum, renders $f(x)$ a maximum or a minimum, a being greater than unity.

(*c*) Any value of x that renders $f(x)$ a maximum or a minimum, renders $\frac{1}{f(x)}$ a minimum or a maximum.

(*d*) Any value of x that renders $c + f(x)$ a maximum or a minimum, renders $f(x)$ a maximum or a minimum.

(*e*) Any value of x that renders $f(x)$ positive, and a maximum or a minimum, renders $[f(x)]^n$ a maximum or a minimum, n being a positive whole number.

EXAMPLES.

1. Find what values of x render $4x^3-15x^2+12x-1$ a maximum or a minimum.

Here $f(x)=4x^3-15x^2+12x-1$;

$$\therefore f'(x)=12x^2-30x+12, \text{ and } f''(x)=24x-30;$$

hence the roots of $f'(x)=12x^2-30x+12=0$, which are $\frac{1}{2}$ and 2, are the *critical* values of x (§ 111).

But $f''(\frac{1}{2})=[24x-30]_{\frac{1}{2}}=-18$,

and $f''(2)=[24x-30]_2=+18$;

hence when $x=\frac{1}{2}$, the function is a maximum (§ 112), and when $x=2$, it is a minimum.

2. Find the maxima and minima of $x^3-9x^2+15x-3$.

Here $f'(x)=3x^2-18x+15$, and $f''(x)=6x-18$;

therefore 5 and 1, the roots of $f'(x)=3x^2-18x+15=0$, are the critical values (§ 111).

$$f''(5)=[6x-18]_5=+12, \text{ and } f''(1)=-12;$$

$$\therefore f(5), \text{ or } [x^3-9x^2+15x-3]_5, [=-28] \text{ is a min.},$$

and $f(1)$, or $[x^3-9x^2+15x-3]_1, [=4]$ is a max.

Let the student construct the locus of $y=x^3-9x^2+15x-3$, and thus exhibit these results geometrically.

3. Examine x^3-3x^2+3x+7 for maxima and minima.

Here $f'(x)=3x^2-6x+3$, $f''(x)=6x-6$, and $f'''(x)=6$;

therefore 1 is the critical value.

But $f''(1)=[6x-6]_1=0$, and $f'''(1)=6$;

hence the function has neither a maximum nor a minimum (§ 112).

4. Examine $x^5-5x^4+5x^3-1$ for maxima and minima.

Here $f(x)=x^5-5x^4+5x^3-1$;

$$\therefore f'(x) = 5x^4 - 20x^3 + 15x^2,\ f''(x) = 20x^3 - 60x^2 + 30x,$$

and $f'''(x) = 60x^2 - 120x + 30$;

therefore 1, 3, and 0 are the critical values.

Since $f''(1) = -10$, $f(1)[=0]$ is a maximum.

Since $f''(3) = +90$, $f(3)[=-28]$ is a minimum.

Since $f''(0) = 0$, and $f'''(0) = 30$,

$f(0)$ is neither a maximum nor a minimum.

5. Examine $(x-1)^4(x+2)^3$ for maxima and minima.

Here $f'(x) = (x-1)^3(x+2)^2(7x+5)$,

and the critical values are -2, $-\frac{5}{7}$, and $+1$.

In this example, the first method is to be preferred. By inspection, we see that

$f'(-2-h)$ and $f'(-2+h)$ are both positive;

hence $f(-2)$ is neither a maximum nor a minimum (§ 112).

$f'(-\frac{5}{7}-h)$ is $+$, and $f'(-\frac{5}{7}+h)$ is $-$;

hence $f(-\frac{5}{7})$ is a maximum (§ 112).

$f'(1-h)$ is $-$, and $f'(1+h)$ is $+$;

hence $f(1)[=0]$ is a minimum.

6. Examine $b + c(x-a)^{\frac{2}{3}}$ for maxima and minima.

Here $$f'(x) = \frac{2c}{3(x-a)^{\frac{1}{3}}};$$

and the critical value is a, the root of $f'(x) = \infty$.

$$\frac{2c}{3(a-h-a)^{\frac{1}{3}}} \text{ is } -, \text{ and } \frac{2c}{3(a+h-a)^{\frac{1}{3}}} \text{ is } +;$$

$\therefore f(a)[=b]$ is a minimum.

7. Examine $\dfrac{(a-x)^3}{a-2x}$ for maxima and minima.

Here $$f'(x) = \frac{(a-x)^2(4x-a)}{(a-2x)^2};$$

and the equations $f'(x) = 0$ and $f'(x) = \infty$ give $\frac{a}{4}$, $\frac{a}{2}$, and a as the critical values.

By inspection, we see that $f'(x)$ changes from negative to positive when $x = \frac{a}{4}$; hence $f\left(\frac{a}{4}\right)$ is a minimum. But, as $f'(x)$ does not change its sign when $x = a$ or $\frac{a}{2}$, $f(a)$ and $f\left(\frac{a}{2}\right)$ are neither maxima nor minima.

8. Examine $c + \sqrt{4a^2x^2 - 2ax^3}$ for maxima and minima.

By (d), (e), and (a) of § 114, any value of x that renders $c + \sqrt{4a^2x^2 - 2ax^3}$ a maximum or a minimum, renders

$$\sqrt{4a^2x^2 - 2ax^3},\ 4a^2x^2 - 2ax^3, \text{ and } 2ax^2 - x^3$$

a maximum or a minimum.

Hence, let $f(x) = 2ax^2 - x^3$, etc.

Ans. When $x = 0$, $c + \sqrt{4a^2x^2 - 2ax^3}$ is a minimum;
" $x = \frac{4}{3}a$, " " is a maximum.

9. What values of x render $2x^3 - 21x^2 + 36x - 20$ a maximum or a minimum?

Ans. $f(1)$ is a max.; $f(6)$ is a min.

10. Examine $3x^5 - 125x^3 + 2160x$ for maxima and minima.

Ans. $f(-4)$ and $f(3)$ are max.; $f(-3)$ and $f(4)$ are min.

11. Examine $x^3 - 3x^2 + 6x + 7$ for maxima and minima.

Ans. It has neither a max. nor a min.

12. If $f'(x) = x^3(x-1)^2(x-2)^3(x-3)^4$, what values of x render $f(x)$ a maximum or a minimum?

Ans. $f(0)$ is a max.; $f(2)$ is a min.

13. Examine $x(x+a)^2(a-x)^3$ for maxima and minima.

Ans. $f(-a)$ and $f\left(\frac{a}{3}\right)$ are max.; $f\left(-\frac{a}{2}\right)$ is a min.

14. Examine $\dfrac{x^2-7x+6}{x-10}$ for maxima and minima.

Ans. $f(4)$ is a max.; $f(16)$ is a min.

15. Examine $\dfrac{(x+2)^3}{(x-3)^2}$ for maxima and minima.

Ans. $f(3)$ is a max.; $f(13)$ is a min.

16. Prove that $\sin x+\cos x$ is a maximum when $x=\dfrac{\pi}{4}$.

17. Examine $\dfrac{x}{\log x}$ for maxima and minima.

Ans. $\dfrac{e}{\log e}$ is a min.

18. Prove that $x^{\frac{1}{x}}$ is a maximum when $x=e$.

19. Prove that $\sin x(1+\cos x)$ is a maximum when $x=\dfrac{\pi}{3}$.

20. Prove that $\dfrac{x}{1+x\tan x}$ is a maximum when $x=\cos x$.

21. Examine the curve $y=x^3-3x^2-24x+85$ for maxima and minima ordinates.

Ans. 113 is a max.; 5 is a min.

22. Examine $y=x^3-9x^2+24x+16$ for maxima and minima ordinates.

Ans. 36 is a max.; 32 is a min.

23. Examine $y=x^3-3x^2-9x+5$ for maxima and minima ordinates.

Ans. 10 is a max.; -22 is a min.

24. Examine $y=x^5-5x^4+5x^3+1$ for maxima and minima ordinates.

Ans. 2 is a max.; -26 is a min.

25. Examine $y=\sin^3 x\cos x$ for maxima and minima ordinates.

Ans. When $x=\frac{1}{3}\pi$, $y=\frac{3}{16}\sqrt{3}$, a max.

Geometric Problems.

1. Find the altitude of the maximum cylinder that can be inscribed in a given right cone.

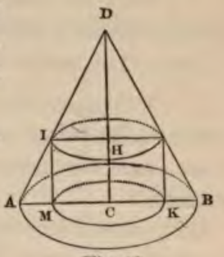

Fig. 28.

Let IK be the cylinder inscribed in the given cone DAB. Let $a = \text{DC}$, $b = \text{AC}$, $y = \text{MC}$, $x = \text{IM}$, and $V =$ the volume of the cylinder; then $V = \pi x y^2$.

From the similar triangles ADC and IDH, we find

$$y = \frac{b}{a}(a - x);$$

$$\therefore V = \pi \frac{b^2}{a^2} x (a - x)^2,$$

which is the function whose maximum is required.

Let $\quad f(x) = x(a - x)^2$, etc. § 114, (*a*).

Ans. The altitude of the cylinder $= \frac{1}{3}$ that of the cone.

2. Find the altitude of the maximum cone that can be inscribed in a sphere whose radius is r.

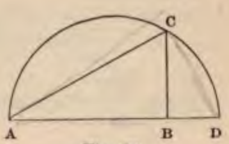

Fig. 29.

Let ACD and ACB be the semicircle and the triangle which generate the sphere and the cone. Let $x = \text{AB}$, $y = \text{BC}$, and $V =$ the volume of the cone; then $V = \frac{1}{3}\pi x y^2$.

Since $\quad y^2 = \text{AB} \cdot \text{BD} = x(2r - x)$,

$$V = \tfrac{1}{3}\pi x^2 (2r - x),$$

which is the function whose maximum is required.

Ans. The altitude of the cone $= \frac{4}{3}$ the radius of the sphere.

3. Find the altitude of the maximum cylinder that can be inscribed in a sphere whose radius is r.

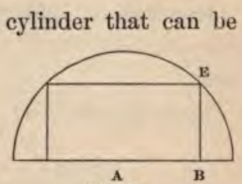

Fig. 30.

Let $\quad x = \text{AB}$, and $y = \text{BE}$;

then $\quad V = 2\pi x y^2 = 2\pi x (r^2 - x^2)$.

Ans. Altitude $= \frac{2}{3} r \sqrt{3}$.

4. Find the maximum rectangle that can be inscribed in an ellipse whose semi-axes are a and b.

Ans. The sides are $a\sqrt{2}$ and $b\sqrt{2}$; the area $= 2ab$.

5. Find the maximum cylinder that can be inscribed in an oblate spheroid whose semi-axes are a and b.

Ans. The radius of the base $= \frac{1}{3}a\sqrt{6}$; the altitude $= \frac{2}{3}b\sqrt{3}$.

6. The capacity of a closed cylindrical vessel being c, a constant, what is the ratio of its altitude to the diameter of its base, when its entire inner surface is a minimum? What is its altitude?

Let y equal the radius of the base, x the altitude, and S the entire inner surface; then

$$c = \pi x y^2, \tag{1}$$

and

$$S = 2\pi y^2 + 2\pi y x. \tag{2}$$

From (1),

$$\frac{dy}{dx} = -\frac{y}{2x}. \tag{3}$$

From (2),

$$\frac{dS}{dx} = 4\pi y\frac{dy}{dx} + 2\pi x\frac{dy}{dx} + 2\pi y. \tag{4}$$

Since $\frac{dS}{dx} = 0$ when S is a minimum, from (4) we have

$$\frac{dy}{dx} = -\frac{y}{2y+x}. \tag{5}$$

From (3) and (5),

$$-\frac{y}{2x} = \frac{-y}{2y+x}, \text{ or } x = 2y. \tag{6}$$

Hence, as S evidently has a minimum value, it is a minimum when the altitude of the cylinder is equal to the diameter of its base.

From (1) and (6),

$$x = 2\sqrt[3]{\frac{c}{2\pi}}.$$

This problem might have been solved like those preceding it; that is, by eliminating y between (1) and (2) at first. In many problems, however, the method given in this example is much to be preferred.

7. The capacity of a cylindrical vessel with open top being constant, what is the ratio of its altitude to the radius of its base when its inner surface is a minimum?

Ans. Its altitude = the radius of its base.

8. A square piece of sheet lead has a square cut out at each corner; find the side of the square cut out when the remainder of the sheet will form a vessel of maximum capacity.

Ans. A side = $\frac{1}{6}$ the edge of the sheet of lead.

9. Find the arc of the sector that must be cut from a circular piece of paper, that the remaining sector may form the convex surface of a cone of maximum volume, r being the radius of the circle.

Ans. The arc $= 2\pi r(1 - \frac{1}{3}\sqrt{6})$.

10. A person, being in a boat 3 miles from the nearest point of the beach, wishes to reach in the shortest time a place 5 miles from that point along the shore; supposing he can walk 5 miles an hour, but row only at the rate of 4 miles an hour, required the place where he must land.

Ans. 1 mile from the place to be reached.

11. Find the maximum right cone that can be inscribed in a given right cone, the vertex of the required cone being at the centre of the base of the given cone.

Ans. The ratio of their altitudes is $\frac{1}{3}$.

12. A Norman window consists of a rectangle surmounted by a semicircle. Given the perimeter, required the height and the breadth of the window when the quantity of light admitted is a maximum.

Ans. The radius of the semicircle = the height of the rectangle.

13. Prove that, of all circular sectors having the same perimeter, the sector of maximum area is that in which the circular arc is double the radius.

14. Find the maximum convex surface of a cylinder inscribed in a cone whose altitude is b, and the radius of whose base is a.

Ans. Maximum surface $= \frac{1}{2}\pi ab$.

15. Find the altitude of the cylinder of maximum convex surface that can be inscribed in a given sphere whose radius is r.

Ans. Altitude $= r\sqrt{2}$.

16. Find the altitude of the cone of maximum convex surface that can be inscribed in a given sphere whose radius is r.

Ans. Altitude $= \frac{4}{3}r$.

17. Find the altitude of the parabola of maximum area that can be cut from a given right circular cone.

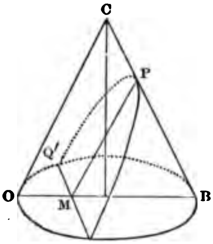

Fig. 31.

Let $\mathrm{OB} = 2b$, $\mathrm{OC} = a$, and $\mathrm{BM} = x$; then

$$\mathrm{QQ'} = 2\,\mathrm{MQ} = 2\sqrt{\mathrm{MB}\cdot\mathrm{MO}}$$
$$= 2\sqrt{x(2b - x)}.$$

Also, $\mathrm{BO} : \mathrm{BM} :: \mathrm{OC} : \mathrm{MP}$,

or $2b : x :: a : \mathrm{MP}$;

$$\therefore \mathrm{MP} = \frac{ax}{2b}.$$

$$\therefore \text{area} = \tfrac{2}{3}\mathrm{QQ'}\cdot\mathrm{MP} = \frac{2a}{3b}\sqrt{x^3(2b - x)}. \qquad \text{§ 66, Ex. 1.}$$

Let $f(x) = x^3(2b - x)$, etc. § 114.

Ans. The parabola is a maximum when its altitude MP is $\frac{3}{4}$ the slant height of the cone.

18. What is the altitude of the maximum cylinder that can be inscribed in a given prolate spheroid; that is, in a solid generated by the revolution of a given ellipse about its major axis?

Ans. Altitude = the major axis divided by $\sqrt{3}$.

19. Find the number of equal parts into which a given number a must be divided that their continued product may be a maximum.

Ans. The number of parts $= \dfrac{a}{e}$, and each part $= e$.

20. A privateer has to pass between two lights A and B, on opposite headlands. The intensity of each light is known, and also the distance between them. At what point must the privateer cross the line joining the lights, so as to be in the light as little as possible?

Let $d =$ the distance AB, and x the distance from A of any point P on AB. Let a and b be the intensities of the lights A and B respectively, at a unit's distance. By a principle of Optics, the intensity of a light at any point equals its intensity at a unit's distance divided by the square of the distance of the point from the light.

Hence $\dfrac{a}{x^2} + \dfrac{b}{(d-x)^2}$ is the function whose minimum we seek.

Ans. $x = \dfrac{da^{\frac{1}{3}}}{a^{\frac{1}{3}} + b^{\frac{1}{3}}}$.

21. The flame of a lamp is directly over the centre of a circle whose radius is r; what is the distance of the flame above the centre when the circumference is illuminated as much as possible?

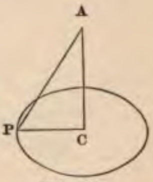

Fig. 32.

Let A be the flame, P any point on the circumference, and $x =$ AC. By a principle of Optics, the intensity of illumination at P varies directly as sin CPA, and inversely as the square of PA.

Hence $\dfrac{ax}{(r^2+x^2)^{\frac{3}{2}}}$ is the function whose maximum is required, in which r is the radius of the circle, and a is the intensity of illumination at a unit's distance from the flame.

Ans. $x = \frac{1}{2} r\sqrt{2}$.

22. On the line joining the centres of two spheres, find the point from which the maximum of spherical surface is visible.

Let $cp = r$, $CP = R$, $cC = d$, and $cA = x$, A being any point on mM. From A draw the tangents Ap and AP; then the sum of the zones whose altitudes are nm and NM is the function whose maximum is required.

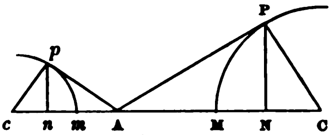

Fig. 33.

Since $cn = \dfrac{r^2}{x}$, by Geometry we have

$$\text{zone } nm = 2\pi r \cdot nm = 2\pi r(r - cn) = 2\pi\left(r^2 - \frac{r^3}{x}\right).$$

Hence $2\pi\left[r^2 + R^2 - \left(\dfrac{r^3}{x} + \dfrac{R^3}{d - x}\right)\right]$ is the function whose maximum is sought.

Ans. $x = \dfrac{dr^{\frac{3}{2}}}{r^{\frac{3}{2}} + R^{\frac{3}{2}}}$.

23. Assuming that the work of driving a steamer through the water varies as the cube of her speed, find her most economical rate per hour against a current running c miles per hour.

Let $v =$ the speed of the steamer in miles per hour.

Then $av^3 =$ the work per hour, a being constant;

and $v - c =$ the actual distance advanced per hour.

Hence $\dfrac{av^3}{v - c} =$ the work per mile of actual advance.

Ans. $v = \frac{3}{2}c$.

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CHAPTER IX.

FUNCTIONS OF TWO OR MORE VARIABLES, AND CHANGE OF THE INDEPENDENT VARIABLE.

115. Functions of two or more Variables. Since any independent variable is some arbitrary function of t, t representing time, a function of any number of independent variables may be regarded as a function of the single variable t, and therefore differentiated by the rules already established.

$f(x, y)$, read "function of x and y," represents any function of x and y; as, $x^3 + xy^2 + xy$ and $\sin(x + y)$.

$f(x, y, z)$, read "function of x, y, and z," represents any function of x, y, and z.

116. A Partial Differential of a function of two or more variables is the differential obtained on the hypothesis that only one of the variables changes.

117. A Total Differential of a function of two or more variables is the differential obtained on the hypothesis that all its variables change.

118. A Partial Derivative of a function of two or more variables is the ratio of the partial differential of the function to the differential of the variable that is supposed to change.

119. A Total Derivative of a function of two or more variables, of which only one is independent, is the ratio of the total differential of the function to the differential of the independent variable.

If $u = f(x, y)$, the partial differentials of u with respect to x and y are written $d_x u$ and $d_y u$, or $\frac{du}{dx}dx$ and $\frac{du}{dy}dy$; and the partial derivatives, or differential coefficients, are written $\frac{d_xu}{dx}$ and $\frac{d_yu}{dy}$, or simply $\frac{du}{dx}$ and $\frac{du}{dy}$.

120. *The total differential of a function of two or more variables is equal to the sum of its partial differentials.*

For, if $u = f(x, y)$, it is evident, from the general principles of differentiation, that du can contain only such terms as are of the first degree in dx and dy.

Hence $$du = \phi(x, y)dx + \phi_1(x, y)dy, \tag{1}$$

in which $\phi(x, y)$ and $\phi_1(x, y)$ represent the sums of the coefficients of dx and dy in the different terms of du.

Let x' and y' be any set of values of x and y; then we have

$$du = \phi(x', y')dx + \phi_1(x', y')dy. \tag{2}$$

Let y be regarded as constant; then $dy = 0$, $\phi(x', y')$ remains unchanged, and (2) becomes

$$d_xu = \phi(x', y')dx. \tag{3}$$

If x is constant, (2) becomes

$$d_yu = \phi_1(x', y')dy. \tag{4}$$

Adding (3) and (4), and remembering that x' and y' are any values of x and y, we have in general

$$d_xu + d_yu = \phi(x, y)dx + \phi_1(x, y)dy = du.$$

Since a similar process of reasoning could be extended to a function of n variables, the theorem is proved.

If x and y were not independent, the demonstration given above would still hold; for the idea of a partial differential of a function sets aside any question concerning the dependence of its variables.

121. Signification of Partial Derivatives. From § 31 it is evident that a partial derivative expresses the ratio of the rate of change of the function to that of its variable, so far as its rate depends on the variable supposed to change; and that a function is an increasing or a decreasing function of any one of its variables, according as its partial derivative with respect to that variable is positive or negative.

EXAMPLES.

1. $u = by^2x + cx^2 + gy^3 + ex$; find du.

Here $d_xu = (by^2 + 2cx + e)dx$,

and $d_yu = (2byx + 3gy^2)dy$;

$$\therefore du = (by^2 + 2cx + e)dx + (2byx + 3gy^2)dy.$$

2. $u = y^x$. *Ans.* $du = y^x \log y\, dx + xy^{x-1}dy.$

3. $u = \log x^y$. $du = \dfrac{y}{x}dx + \log x\, dy.$

4. $u = \tan^{-1}\dfrac{y}{x}$. $du = \dfrac{x\,dy - y\,dx}{x^2 + y^2}.$

5. $u = y^{\sin x}$. $du = y^{\sin x} \log y \cos x\, dx + \dfrac{\sin x}{y^{\text{covers}\, x}}dy.$

6. $u = \log \tan^{-1}\dfrac{x}{y}$. $du = \dfrac{y\,dx - x\,dy}{(x^2 + y^2)\tan^{-1}\dfrac{x}{y}}.$

7. $u = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$. $du = \dfrac{2x}{a^2}dx + \dfrac{2y}{b^2}dy.$

122. If $u = f(x, y, z)$, and $y = \phi(x)$, and $z = \phi_1(x)$, u is *directly* a function of x, and *indirectly* a function of x *through* y *and* z. If $u = f(z, y)$, and $y = \phi(x)$, and $z = \phi_1(x)$, u is, in like manner, indirectly a function of x through y and z. In all such cases the total derivative of u with respect to x can be obtained by finding the value of u in terms of x, and differentiating the result; but in many cases it is more readily obtained by using the formulas of the next article.

123. If $u = f(x, y, z)$, and $y = \phi(x)$, and $z = \phi_1(x)$,

we have $du = \frac{du}{dx}dx + \frac{du}{dy}dy + \frac{du}{dz}dz;$ § 120

$$\therefore \left[\frac{du}{dx}\right]^* = \frac{du}{dx} + \frac{du}{dy}\frac{dy}{dx} + \frac{du}{dz}\frac{dz}{dx},$$

in which $\frac{du}{dx}$, $\frac{du}{dy}$, *and* $\frac{du}{dz}$ *are the partial derivatives, and* $\left[\frac{du}{dx}\right]$ *the total derivative of* u.

COR. 1. If $u = f(x, y)$, and $y = \phi(x)$, $du = \frac{du}{dx}dx + \frac{du}{dy}dy;$

$$\therefore \left[\frac{du}{dx}\right] = \frac{du}{dx} + \frac{du}{dy}\frac{dy}{dx}.$$

COR. 2. If $u = f(y, z)$, and $y = \phi(x)$, and $z = \phi_1(x)$,

$$du = \frac{du}{dy}dy + \frac{du}{dz}dz;$$

$$\therefore \left[\frac{du}{dx}\right] = \frac{du}{dy}\frac{dy}{dx} + \frac{du}{dz}\frac{dz}{dx}.$$

COR. 3. If $u = f(y)$, and $y = \phi(x)$, $du = \frac{du}{dy}dy;$

$$\therefore \frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx}.$$

REM. To make the above theorem and corollaries intelligible, the signification of each term and factor must be had clearly in mind. Thus, in the theorem, $\left[\frac{du}{dx}\right]$ denotes the *total derivative* of u with respect to x, that is, the derivative obtained on the hypothesis that x, y, and z are all changing according to the given conditions; while $\frac{du}{dx}$ denotes the partial derivative of u with respect to x, that is, the derivative obtained on the hypothesis that y and z are constant.

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[*] Analysts are not agreed as to the best means of distinguishing total from partial derivatives; but we shall always distinguish the total derivative of a function of two or more variables by enclosing it in brackets.

EXAMPLES.

1. $u = z^2 + y^3 + zy$, $z = \sin x$, and $y = e^x$; find $\left[\frac{du}{dx}\right]$.

Here
$$\left[\frac{du}{dx}\right] = \frac{du}{dy}\frac{dy}{dx} + \frac{du}{dz}\frac{dz}{dx}. \qquad (1)$$

$$\frac{du}{dy} = 3y^2 + z, \quad \frac{du}{dz} = 2z + y,$$

$$\frac{dz}{dx} = \cos x, \text{ and } \frac{dy}{dx} = e^x.$$

Substituting these values in (1), we have

$$\begin{aligned}\left[\frac{du}{dx}\right] &= (3y^2 + z)e^x + (2z + y)\cos x \\ &= (3e^{2x} + \sin x)e^x + (2\sin x + e^x)\cos x \\ &= 3e^{3x} + e^x(\sin x + \cos x) + \sin 2x.\end{aligned}$$

This same result could be obtained by substituting in u the values of y and z in terms of x, and then differentiating.

2. $u = \tan^{-1}(xy)$, and $y = e^x$. *Ans.* $\left[\frac{du}{dx}\right] = \frac{e^x(1+x)}{1 + x^2e^{2x}}$.

3. $u = e^{ax}(y - z)$, $y = a\sin x$, and $z = \cos x$.

$$\left[\frac{du}{dx}\right] = (a^2 + 1)e^{ax}\sin x.$$

4. $u = \tan^{-1}\frac{y}{x}$, and $x^2 + y^2 = r^2$.

$$\left[\frac{du}{dx}\right] = -\frac{1}{y} \text{ or } -\frac{1}{\sqrt{r^2 - x^2}}.$$

5. $u = \sin\frac{z}{y}$, $z = e^x$, and $y = x^2$.

$$\left[\frac{du}{dx}\right] = (x - 2)\frac{e^x}{x^3}\cos\frac{e^x}{x^2}.$$

6. $u = \sqrt{x^2 + y^2}$, and $y = mx + c$.

$$\left[\frac{du}{dx}\right] = \frac{(1 + m^2)x + mc}{\sqrt{x^2 + (mx + c)^2}}.$$

7. $u = \sin^{-1}(y - z)$, $y = 3x$, and $z = 4x^3$. $\left[\frac{du}{dx}\right] = \frac{3}{\sqrt{1 - x^2}}$.

8. $u = x^4y^2 - \frac{x^4y}{2} + x^4$, and $y = \log x$.

$$\left[\frac{du}{dx}\right] = x^3[4(\log x)^2 + 3\tfrac{1}{2}].$$

124. Implicit Functions. The equation $f(x, y) = 0$ indicates that either y or x is an implicit function of the other. It represents any equation containing x and y when all its terms are in the first member. Implicit functions have been differentiated heretofore, but the formula in § 125 is often useful in obtaining the first derivative of any such function.

125. If $u = f(x, y)$, $du = \frac{du}{dy}dy + \frac{du}{dx}dx$. (1)

If u *is constant,* du = 0, *and from* (1) *we obtain*

$$\frac{dy}{dx} = -\frac{\frac{du}{dx}}{\frac{du}{dy}},$$

in which $\frac{\mathrm{du}}{\mathrm{dx}}$ *and* $\frac{\mathrm{du}}{\mathrm{dy}}$ *are partial derivatives.*

When u is constant, $du = d_xu + d_yu = 0$; but, in general, neither d_xu nor d_yu is zero.

EXAMPLES.

1. $y^3 - 2x^2y + bx = 0$; find $\frac{dy}{dx}$.

Here $\frac{du}{dx} = -4yx + b$, $\frac{du}{dy} = 3y^2 - 2x^2$;

$$\therefore \frac{dy}{dx} = -\frac{\frac{du}{dx}}{\frac{du}{dy}} = \frac{4yx - b}{3y^2 - 2x^2}.$$

2. $x^3 + y^3 - 3axy = c = u.$ $\qquad \dfrac{dy}{dx} = \dfrac{x^2 - ay}{ax - y^2}.$

3. $\dfrac{x^m}{a^m} + \dfrac{y^m}{b^m} = 1.$ $\qquad \dfrac{dy}{dx} = -\left(\dfrac{x}{y}\right)^{m-1}\left(\dfrac{b}{a}\right)^m.$

4. $x \log y - y \log x = 0.$ $\qquad \dfrac{dy}{dx} = \dfrac{y}{x}\left(\dfrac{x \log y - y}{y \log x - x}\right).$

5. $f(ax + by) = c.$

Here $\quad f(ax + by) = c = u;$

$$\therefore \frac{du}{dx} = f'(ax + by)\, a, \quad \frac{du}{dy} = f'(ax + by)\, b;$$

$$\therefore \frac{dy}{dx} = -\frac{a}{b}.$$

6. $x^y - y^x = 0.$ $\qquad \dfrac{dy}{dx} = \dfrac{y^2 - xy \log y}{x^2 - xy \log x}.$

7. $x^3 + 3axy = -y^3.$ $\qquad \dfrac{dy}{dx} = -\dfrac{x^2 + ay}{y^2 + ax}.$

126. General formulas for the successive derivatives of an implicit function can be easily deduced; but they are so complicated that, in practice, it is generally more convenient to differentiate directly the first derivative to obtain the second, and so on, than to use these formulas.

EXAMPLES.

1. Find $\dfrac{d^2y}{dx^2}$ when $y^2 - 2xy + c = 0.$

Here $\quad \dfrac{dy}{dx} = \dfrac{y}{y - x}. \qquad (1)$

Differentiating (1), we have

$$\frac{d^2y}{dx^2} = \frac{(y - x)\dfrac{dy}{dx} - y\left(\dfrac{dy}{dx} - 1\right)}{(y - x)^2} = \frac{y - x\dfrac{dy}{dx}}{(y - x)^2}. \qquad (2)$$

From (1), (2), and the given equation, we obtain

$$\frac{d^2y}{dx^2}=\frac{y^2-2xy}{(y-x)^3}=\frac{-c}{(y-x)^3}.$$

2. Given $y^2-2axy+x^2-c=0$, to find $\dfrac{d^2y}{dx^2}$.

$$\frac{d^2y}{dx^2}=\frac{(a^2-1)(y^2-2axy+x^2)}{(y-ax)^3}=\frac{c(a^2-1)}{(y-ax)^3}.$$

3. Given $y^3+x^3-3axy=0$; show that $\dfrac{d^2y}{dx^2}=-\dfrac{2a^3xy}{(y^2-ax)^3}$.

127. Successive Partial Differentials and Derivatives. If $u=f(x,y)$, x and y being independent, d_xu and d_yu are, in general, functions of both x and y. Differentiating d_xu and d_yu with respect to either variable, we obtain a class of second partial differentials. By differentiating these second partial differentials, we obtain a class of third partial differentials; and so on. In finding these successive partial differentials, we regard dx and dy as constant, since we may suppose x and y to change uniformly.

The successive partial differentials are represented as follows:

$$d_x\left(\frac{du}{dx}dx\right)\left[=\frac{d}{dx}\left(\frac{du}{dx}dx\right)dx\right]=\frac{d^2u}{dx^2}dx^2;$$

$$d_y\left(\frac{du}{dx}dx\right)\left[=\frac{d}{dy}\left(\frac{du}{dx}dx\right)dy\right]=\frac{d^2u}{dx\,dy}dx\,dy;$$

$$d_x\left(\frac{d^2u}{dy^2}dy^2\right)\left[=\frac{d}{dx}\left(\frac{d^2u}{dy^2}dy^2\right)dx\right]=\frac{d^3u}{dy^2dx}dy^2dx;$$

$$d_y\left(\frac{d^2u}{dy^2}dy^2\right)\left[=\frac{d}{dy}\left(\frac{d^2u}{dy^2}dy^2\right)dy\right]=\frac{d^3u}{dy^3}dy^3;\ \text{etc.}$$

Hence $\dfrac{d^3u}{dx\,dy^2}dx\,dy^2$ is a symbol for the result obtained by differentiating $u=f(x,y)$ three times in succession: first, once with respect to x, and then twice with respect to y.

The symbols for the partial derivatives are

$$\frac{d^2u}{dx^2},\ \frac{d^2u}{dx\,dy},\ \frac{d^2u}{dy^2},\ \frac{d^3u}{dx^3},\ \frac{d^3u}{dx^2dy},\ \text{etc.}$$

128. If $u=f(x, y)$, $\dfrac{d^2u}{dy\,dx}=\dfrac{d^2u}{dx\,dy}$, $\dfrac{d^3u}{dy\,dx^2}=\dfrac{d^3u}{dx^2dy}$, etc.;

that is, *if* u *be differentiated* m *times with respect to* x, *and* n *times with respect to* y, *the result is the same, whatever be the order of the differentiations.*

For $$\frac{du}{dx}=\underset{\Delta x\doteq 0}{\text{limit}}\left[\frac{f(x+\Delta x, y)-f(x, y)}{\Delta x}\right].$$

Regarding this expression as a function of y, finding its increment, dividing by Δy, and remembering that the difference between the limits of two variables is equal to the limit of their difference, we have

$$\frac{d}{dy}\left(\frac{du}{dx}\right)$$
$$=\underset{\Delta y\doteq 0}{\text{limit}}\left\{\frac{\underset{\Delta x\doteq 0}{\text{limit}}\left[\dfrac{f(x+\Delta x, y+\Delta y)-f(x, y+\Delta y)-f(x+\Delta x, y)+f(x, y)}{\Delta x}\right]}{\Delta y}\right\}.$$

In like manner, we should obtain for the value of $\dfrac{d}{dx}\left(\dfrac{du}{dy}\right)$ the same expression, except that the order of the limits would be reversed; but, from the nature of the process of passing to a limit, it is evident that this change of order does not affect the value of the expression.

Hence $$\frac{d}{dy}\left(\frac{du}{dx}\right)=\frac{d}{dx}\left(\frac{du}{dy}\right),\ \text{or}\ \frac{d^2u}{dx\,dy}=\frac{d^2u}{dy\,dx}. \tag{1}$$

Again, since

$$\frac{d^2u}{dx\,dy}=\frac{d^2u}{dy\,dx},$$

$$\frac{d^3u}{dx\,dy\,dx}=\frac{d^3u}{dy\,dx^2}. \tag{2}$$

But, from the principle in (1), we have

$$\frac{d}{dx}\cdot\frac{d}{dy}\left(\frac{du}{dx}\right)=\frac{d}{dy}\cdot\frac{d}{dx}\left(\frac{du}{dx}\right),\text{ or }\frac{d^3u}{dx\,dy\,dx}=\frac{d^3u}{dx^2dy}\cdot \quad (3)$$

From (2) and (3),

$$\frac{d^3u}{dy\,dx^2}=\frac{d^3u}{dx^2dy}.$$

This method of reasoning evidently applying to all cases, the theorem is established.

Examples.

1. $u=\cos(x+y)$; verify $\frac{d^2u}{dy\,dx}=\frac{d^2u}{dx\,dy}$.

2. $u=x^3y^2+ay^3$; verify $\frac{d^3u}{dy^2dx}=\frac{d^3u}{dx\,dy^2}$.

3. $u=\log(x+y)$; find the first, second, and third partial derivatives.

4. $u=\tan^{-1}\frac{y}{x}$; verify $\frac{d^3u}{dx^2dy}=\frac{d^3u}{dy\,dx^2}$.

5. $u=\sin(bx^5+ay^5)$; verify $\frac{d^3u}{dy^2dx}=\frac{d^3u}{dx\,dy^2}$.

129. *To find the successive differentials of a function of two independent variables.*

Let $u=f(x,y)$; then

$$du=\frac{du}{dx}dx+\frac{du}{dy}dy. \quad (1)$$

Differentiating (1), remembering that $\frac{du}{dx}$ and $\frac{du}{dy}$ are, in general, functions of both x and y; and that, as x and y are independent, dx and dy may be regarded as constant, we have

$$d^2u=d_x\left(\frac{du}{dx}dx\right)+d_y\left(\frac{du}{dx}dx\right)+d_x\left(\frac{du}{dy}dy\right)+d_y\left(\frac{du}{dy}dy\right)$$

$$=\frac{d^2u}{dx^2}dx^2+\frac{d^2u}{dx\,dy}dx\,dy+\frac{d^2u}{dy\,dx}dy\,dx+\frac{d^2u}{dy^2}dy^2$$

$$=\frac{d^2u}{dx^2}dx^2+2\frac{d^2u}{dx\,dy}dx\,dy+\frac{d^2u}{dy^2}dy^2. \qquad (2)$$

Differentiating (2), remembering that, in general, each term is a function of both x and y; and that the total differential of each is the sum of its partial differentials, we obtain

$$d^3u=\frac{d^3u}{dx^3}dx^3+3\frac{d^3u}{dx^2dy}dx^2dy+3\frac{d^3u}{dx\,dy^2}dx\,dy^2+\frac{d^3u}{dy^3}dy^3.$$

Differentiating this equation, we obtain an analogous expression for d^4u, and so on.

By observing the analogy between the values of d^2u and d^3u, and the development of the second and third powers of a binomial, the formula for d^nu may be easily written out.

130. Change of the Independent Variable. The forms of the successive derivatives of $\frac{dy}{dx}$ used thus far have been obtained upon the hypothesis that x is the independent variable, and that dx is constant. In applications of the Differential Calculus, it is often desirable to change the independent variable, and to regard the original function, or some other variable, as the independent variable. We proceed to find the forms of the successive derivatives of $\frac{dy}{dx}$ upon other hypotheses than that dx is constant.

131. *To find the successive derivatives of* $\frac{dy}{dx}$; *that is, the forms of* $\frac{d}{dx}\left(\frac{dy}{dx}\right)$, $\frac{d}{dx}\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right)$, *etc., when neither* x *nor* y *is independent.*

If neither x nor y is independent, $\frac{dy}{dx}$ is a fraction with a variable numerator and denominator; and we have

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{dx\,d^2y-dy\,d^2x}{dx^3}. \qquad (1)$$

Differentiating (1), we obtain

$$\frac{d}{dx}\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{dx\,d^2y-dy\,d^2x}{dx^3}\right)$$

$$=\frac{(d^3y\,dx-d^3x\,dy)dx-3(d^2y\,dx-d^2x\,dy)d^2x}{dx^5}. \qquad (2)$$

In like manner, we obtain the other successive derivatives.

COR. 1. If y is independent, that is, if dy is constant,

$d^2y=0$, and $d^3y=0$,

and (1) and (2) become

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=-\frac{d^2x\,dy}{dx^3}, \qquad (3)$$

and

$$\frac{d}{dx}\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{3(d^2x)^2dy-d^3x\,dy\,dx}{dx^5}. \qquad (4)$$

COR. 2. If dx is constant, $d^2x=d^3x=0$, and (1) and (2) become

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}, \text{ and } \frac{d}{dx}\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^3y}{dx^3},$$

which agrees with § 74.

REM. Hence, to transform a differential expression in which x is independent, into its equivalent in which neither x nor y is independent, we replace $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, etc., by their equivalents upon the new hypothesis, which are found in (1), (2), etc.

If, in the transformed expression, a new variable θ, of which x is a function, is to be the independent variable, in the general result obtained above we replace x, dx, d^2x, etc., by their values in terms of θ and its differentials.

If y is to be the independent variable, in the given expression we replace $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, etc., by their equivalents in (3), (4), etc.; or, in the general result first obtained above, we put $d^2y=0$, $d^3y=0$, etc.

EXAMPLES.

1. Given $yd^2y + dy^2 + dx^2 = 0$, in which x is independent, to find the transformed equation in which neither x nor y is independent; also the one in which y is independent.

Dividing both members by dx^2, substituting for $\frac{d^2y}{dx^2}$ the second member of (1), § 131, and multiplying both members by dx^3, we have

$$y(d^2y\,dx - d^2x\,dy) + dy^2\,dx + dx^3 = 0,$$

in which neither x nor y is independent.

Putting $d^2y = 0$, and dividing by $-dy^3$, we have

$$y\frac{d^2x}{dy^2} - \frac{dx^3}{dy^3} - \frac{dx}{dy} = 0,$$

in which the position of dy indicates that y is independent.

2. Given $\frac{d^2y}{dx^2} - \frac{x}{1-x^2}\frac{dy}{dx} + \frac{y}{1-x^2} = 0$, in which x is independent, to find the transformed equation when $x = \cos\theta$, and θ is independent.

Substituting for $\frac{d^2y}{dx^2}$ the second member of (1), we have

$$\frac{d^2y\,dx - d^2x\,dy}{dx^3} - \frac{x}{1-x^2}\frac{dy}{dx} + \frac{y}{1-x^2} = 0. \tag{1}$$

Since $x = \cos\theta,\ 1 - x^2 = \sin^2\theta,$

$$dx = -\sin\theta\,d\theta, \text{ and } d^2x = -\cos\theta\,d\theta^2.$$

Substituting these values in (1), and simplifying, we have

$$\frac{d^2y}{d\theta^2} + y = 0,$$

in which θ is independent.

3. Given $R=\dfrac{\left[1+\dfrac{dy^2}{dx^2}\right]^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}$ in which x is independent, to find the value of R when $x=\rho\cos\theta$, $y=\rho\sin\theta$, and θ is independent.

From (1), § 131,

$$R=\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dx\,d^2y-dy\,d^2x},\tag{1}$$

in which neither x nor y is independent.

From $y=\rho\sin\theta$, and $x=\rho\cos\theta$, we obtain

$$dy=\sin\theta\,d\rho+\rho\cos\theta\,d\theta,$$

$$dx=\cos\theta\,d\rho-\rho\sin\theta\,d\theta,$$

$$d^2y=\sin\theta\,d^2\rho+2\cos\theta\,d\theta\,d\rho-\rho\sin\theta\,d\theta^2,$$

and

$$d^2x=\cos\theta\,d^2\rho-2\sin\theta\,d\theta\,d\rho-\rho\cos\theta\,d\theta^2.$$

Substituting these values in (1), and simplifying, we have

$$R=\frac{\left(\dfrac{d\rho^2}{d\theta^2}+\rho^2\right)^{\frac{3}{2}}}{\rho^2+2\dfrac{d\rho^2}{d\theta^2}-\rho\dfrac{d^2\rho}{d\theta^2}}.$$

4. Given $\dfrac{d^2y}{dx^2}+\dfrac{1}{x}\dfrac{dy}{dx}+y=0$, to find the transformed equation when $x^2=4z$, and z is independent.

Ans. $z\dfrac{d^2y}{dz^2}+\dfrac{dy}{dz}+y=0.$

5. Given $(1-x^2)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}=0$, to find the transformed equation when $x=\cos z$, and z is independent.

Ans. $\dfrac{d^2y}{dz^2}=0.$

6. Given $z=\dfrac{x\,dy-y\,dx}{y\,dy+x\,dx}$, to find the transformed equation when $x=\rho\cos\theta$, $y=\rho\sin\theta$, and ρ is independent.

Ans. $z=\dfrac{\rho d\theta}{d\rho}.$

CHAPTER X.

TANGENTS, NORMALS, AND ASYMPTOTES.

132. The **Rectangular Equation** of the **Tangent** to any plane curve at (x', y') is

$$y - y' = \frac{dy'^{\,*}}{dx'}(x - x'). \qquad (a)$$

For line (a) passes through (x', y'); and, by § 16, it has the slope of the curve at (x', y').

Cor. When the axes are oblique, $\frac{dy'}{dx'}$ is evidently the ratio of the sines of the angles which the curve at (x', y') makes with the axes; hence, in this case also, (a) is the equation of the tangent.

133. The **Rectangular Equation** of the **Normal** to any plane curve at (x', y') is

$$y - y' = -\frac{dx'}{dy'}(x - x'). \qquad (b)$$

For, the axes being rectangular, line (b) is perpendicular to line (a) of § 132, and therefore to the curve, at (x', y').

Examples.

1. Find the equations of the tangent and normal to the parabola $y^2 = 2px$.

Here $\quad \frac{dy}{dx} = \frac{p}{y}; \ \therefore \frac{dy'}{dx'} = \frac{p}{y'}.$

---

* $\frac{dy'}{dx'}$ represents the value of $\frac{dy}{dx}$ at the point (x', y').

This value of $\frac{dy'}{dx'}$ substituted in (a) of § 132, and (b) of § 133, gives

$$y - y' = \frac{p}{y'}(x - x'), \tag{1}$$

and $$y - y' = -\frac{y'}{p}(x - x'),$$

as the equations of the tangent and normal respectively.

Since $y'^2 = 2px'$, equation (1) by reduction becomes

$$yy' = p(x + x').$$

2. Find the equations of the tangent and normal to the circle $x^2 + y^2 = r^2$.

Ans. $yy' + xx' = r^2$; $y = \frac{y'}{x'}x$.

3. Find the equations of the tangent and normal to the ellipse $a^2y^2 + b^2x^2 = a^2b^2$.

Ans. $a^2yy' + b^2xx' = a^2b^2$; $y - y' = \frac{a^2y'}{b^2x'}(x - x')$.

4. Find the equations of the tangent and normal to the hyperbola $a^2y^2 - b^2x^2 = -a^2b^2$.

Ans. $a^2yy' - b^2xx' = -a^2b^2$; $y - y' = -\frac{a^2y'}{b^2x'}(x - x')$.

5. Find the equations of the tangent and normal to the cissoid $y^2 = \frac{x^3}{2a - x}$.

Ans. $y - y' = \pm\frac{x'^{\frac{1}{2}}(3a - x')}{(2a - x')^{\frac{3}{2}}}(x - x')$;

$$y - y' = \mp\frac{(2a - x')^{\frac{3}{2}}}{x'^{\frac{1}{2}}(3a - x')}(x - x').$$

6. Find the equation of the tangent to $y^2 = 2x^2 - x^3$ at $x = 1$.

Ans. $y = \frac{1}{2}x + \frac{1}{2}$; $y = -\frac{1}{2}x - \frac{1}{2}$.

7. Find the equation of the normal to $y^2 = 6x - 5$ at $y = 5$.

Ans. $y = -\frac{5}{3}x + \frac{40}{3}$.

8. Find the equation of the tangent to the hyperbola referred to its asymptotes, $xy = m$.

Ans. $y = -\frac{y'}{x'}x + 2y'$.

9. Find the equation of the tangent to the cycloid

$x = r \operatorname{vers}^{-1}\frac{y}{r} - \sqrt{2ry - y^2}$. *Ans.* $y - y' = \sqrt{\frac{2r - y'}{y'}}\,(x - x')$.

134. Length of Subtangent, Subnormal, Tangent, and Normal.

Let PT be the tangent at the point P (x', y'), and PS the normal. Draw the ordinate PM; then TM is called the *subtangent*, and MS the *subnormal*.

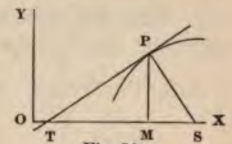

Fig. 34.

$$\text{TM} = \frac{\text{MP}}{\tan \text{MTP}} = y'\frac{dx'}{dy'};$$

$$\therefore \text{subtangent} = y'\frac{dx'}{dy'}. \qquad (1)$$

$$\text{MS} = \text{MP} \tan \text{MPS} = y' \tan \text{MTP} = y'\frac{dy'}{dx'};$$

$$\therefore \text{subnormal} = y'\frac{dy'}{dx'}. \qquad (2)$$

$$\text{PT} = \sqrt{\overline{\text{MP}}^2 + \overline{\text{TM}}^2} = \sqrt{y'^2 + \left(y'\frac{dx'}{dy'}\right)^2};$$

$$\therefore \text{tangent} = y'\sqrt{1 + \left(\frac{dx'}{dy'}\right)^2}. \qquad (3)$$

$$\text{PS} = \sqrt{\overline{\text{MP}}^2 + \overline{\text{MS}}^2} = \sqrt{y'^2 + \left(y'\frac{dy'}{dx'}\right)^2};$$

$$\therefore \text{normal} = y'\sqrt{1 + \left(\frac{dy'}{dx'}\right)^2}. \qquad (4)$$

REM. If the subtangent be reckoned from the point T (Fig. 34), and the subnormal from M, each will be positive or negative according as it extends to the right or left.

EXAMPLES.

1. Find the values of the subtangent and subnormal of the ellipse $a^2y^2 + b^2x^2 = a^2b^2$.

$$\text{Subt.} = y'\frac{dx'}{dy'} = -\frac{a^2y'^2}{b^2x'} = \frac{x'^2 - a^2}{x'};$$

$$\text{Subn.} = y'\frac{dy'}{dx'} = -\frac{b^2x'}{a^2}.$$

2. Find the values of the subtangent and subnormal of the parabola, circle, hyperbola, and cissoid.

Ans. Parabola: $\text{subt.} = 2x'$; $\text{subn.} = p$.

Circle: $\text{subt.} = -\dfrac{y'^2}{x'}$; $\text{subn.} = -x'$.

Hyperbola: $\text{subt.} = \dfrac{x'^2 - a^2}{x'}$; $\text{subn.} = \dfrac{b^2x'}{a^2}$.

Cissoid: $\text{subt.} = \dfrac{x'(2a - x')}{3a - x'}$; $\text{subn.} = \dfrac{x'^2(3a - x')}{(2a - x')^2}$.

3. Find the value of the subtangent of the logarithmic curve $y = a^x$; also of $y^2 = 3x^2 - 12$ at $x = 4$. *Ans.* m; 3.

4. Find the values of the subnormal and normal of the cycloid.

$$\text{Subnormal} = \sqrt{(2r - y)y} = \sqrt{\text{HB}\cdot\text{HD}} = \text{PH} = \text{ED}.$$

$$\text{Normal} = \text{PD} = \sqrt{\overline{\text{ED}}^2 + \overline{\text{EP}}^2} = \sqrt{2ry}.$$

Thus the normal passes through the foot of the vertical diameter of the generating circle, when it is in position for the point to which the normal is drawn. Moreover, since DPB is a right angle, the tangent passes through the other extremity of the vertical diameter. This property furnishes a simple method of drawing a tangent and normal to the cycloid at any

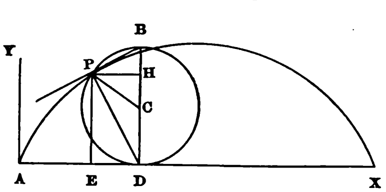

Fig. 35.

point. Thus, to draw a tangent and normal at P, put the generating circle in position for this point, and draw the vertical diameter BD. The lines drawn from B and D through P will be respectively the required tangent and normal.

That PB is tangent to the cycloid at P is further evident; since, when the generating point reaches the position P, it is rotating about the point D, and is therefore moving in a direction perpendicular to DP.

135. Fundamental Principle in the Method of Limits. Let α, α_1, β and β_1, be any four variables, so related that

$$\text{limit}\frac{\alpha}{\alpha_1}=1,\ \text{limit}\frac{\beta}{\beta_1}=1,\ \text{and limit}\frac{\alpha}{\beta}=c;$$

then, since $\frac{\alpha}{\beta}=\frac{\alpha}{\beta}\cdot\frac{\alpha_1}{\alpha_1}\cdot\frac{\beta_1}{\beta_1}=\frac{\alpha_1}{\beta_1}\cdot\frac{\alpha}{\alpha_1}\cdot\frac{\beta_1}{\beta}$,

$$\text{limit}\frac{\alpha}{\beta}=\text{limit}\frac{\alpha_1}{\beta_1}\times\text{limit}\frac{\alpha}{\alpha_1}\times\text{limit}\frac{\beta_1}{\beta}=\text{limit}\frac{\alpha_1}{\beta_1}.$$

Hence, *in any problem concerning the limit of the ratio of two variables, either may be replaced by any other variable, the limit of whose ratio to it is unity.*

136. Length of Subtangent, Subnormal, Tangent, and Normal in Polar Curves. Let OX be the polar axis, and P any point on the curve mn. Let arc PD $=\Delta s$, and OB $=1$; then arc BE $=\Delta\theta$, arc PM $=\rho\Delta\theta$, and MD $=\Delta\rho$. Draw the chords PM and PD, the tangent PZ, and PH perpendicular, and ZH parallel, to OP, thus forming the right triangle PZH; then

$$\underset{\Delta\theta\doteq0}{\text{limit}}\left[\frac{\rho\Delta\theta}{\text{chord PM}}\right]=1,\quad \S 48.$$

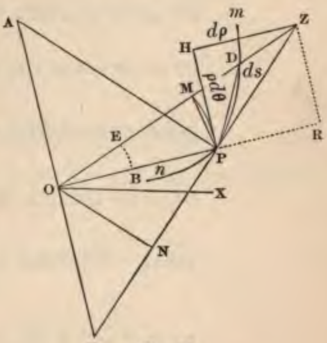

Fig. 36.

and $$\lim_{\Delta s \doteq 0}\left[\frac{\Delta s}{\text{chord PD}}\right] = 1.$$

$$\therefore \lim_{\Delta\theta \doteq 0}\left[\frac{\rho\Delta\theta}{\Delta s}\right] = \lim_{\Delta\theta \doteq 0}\left[\frac{\text{chord PM}}{\text{chord PD}}\right], \qquad \S\ 135.$$

and $$\lim_{\Delta\theta \doteq 0}\left[\frac{\Delta\rho}{\Delta s}\right] = \lim_{\Delta\theta \doteq 0}\left[\frac{\text{MD}}{\text{chord PD}}\right].$$

The limit of angle MPD is evidently HPZ; and, in the isosceles triangle POM, the limit of angle POM being zero, the limit of PMO or its supplement PMD is a right angle. Hence the angles of the triangle HPZ are the limits of the angles of the triangle MPD. Therefore the ratios of the sides of the triangle HPZ equal the limits of the ratios of $\rho\Delta\theta$, Δs, and $\Delta\rho$. Hence these sides may be taken as $\rho\, d\theta$, ds, and $d\rho$.

Draw OT perpendicular to OP, and produce it until it meets the tangent in T. Draw also the normal PA, and the perpendicular ON upon the tangent. The lengths PT and PA are called respectively the *polar tangent* and *polar normal;* OA is the *polar subnormal*, and OT the *polar subtangent.*

$$\tan \text{OPT} = \tan \text{HZP} = \frac{\text{HP}}{\text{HZ}} = \frac{\rho\, d\theta}{d\rho}. \tag{1}$$

$$\sin \text{OPT} = \sin \text{HZP} = \frac{\text{HP}}{\text{ZP}} = \frac{\rho\, d\theta}{ds}. \tag{2}$$

From triangle HPZ,

$$ds^2 = d\rho^2 + \rho^2 d\theta^2. \tag{3}$$

$$\text{OT} = \text{OP} \tan \text{OPT} = \rho^2\frac{d\theta}{d\rho};$$

$$\therefore \text{polar subtangent} = \rho^2\frac{d\theta}{d\rho}. \tag{4}$$

$$\text{OA} = \text{OP} \tan \text{OPA} = \text{OP} \cot \text{OPT} = \frac{d\rho}{d\theta};$$

$$\therefore \text{polar subnormal} = \frac{d\rho}{d\theta}. \tag{5}$$

$$\text{PT} = \sqrt{\overline{\text{OP}}^2 + \overline{\text{OT}}^2} = \sqrt{\rho^2 + \rho^4\frac{d\theta^2}{d\rho^2}};$$

$$\therefore \text{polar tangent} = \rho\sqrt{1+\rho^2\frac{d\theta^2}{d\rho^2}}. \qquad (6)$$

$$\text{AP} = \sqrt{\overline{\text{OP}}^2 + \overline{\text{OA}}^2} = \sqrt{\rho^2 + \frac{d\rho^2}{d\theta^2}};$$

$$\therefore \text{polar normal} = \sqrt{\rho^2 + \frac{d\rho^2}{d\theta^2}}. \qquad (7)$$

$$p = \text{ON} = \text{OP} \sin \text{OPT} = \rho^2\frac{d\theta}{ds} = \frac{\rho^2 d\theta}{\sqrt{\rho^2 d\theta^2 + d\rho^2}}. \qquad (8)$$

137. That the sides of the triangle PZH (Fig. 36) may be taken as ds, $d\rho$, and $\rho\, d\theta$ can be proved also as follows:

When the generatrix of the curve is at P, the radius vector is increasing in length in the direction of PR, and the extremity of the radius vector drawn to P is moving in the direction of PH, at the rates at which the generatrix is then moving in these directions. If, at P, the motion of the generatrix became uniform along the tangent PZ, it is evident that any simultaneous increments of its distances from P and lines PH and OR may be taken as ds, $d\rho$, and the differential of the arc traced by the extremity of the radius vector to P, which equals $\rho\, d\theta$; for, if $\text{OB} = 1$, B describes the measuring arc of the variable angle θ, and P is moving, as OP revolves, ρ times as fast as B. Hence $\text{PH} = \rho\, d\theta$, and $\text{HZ} = d\rho$, if $\text{PZ} = ds$.

EXAMPLES.

1. Find the subtangent, subnormal, tangent, normal, and p, or the length of the perpendicular from the pole to the tangent, of the spiral of Archimedes $\rho = a\theta$.

$$\text{subnormal} = \frac{d\rho}{d\theta} = a;$$

$$\text{subtangent} = \rho^2\frac{d\theta}{d\rho} = \frac{\rho^2}{a};$$

$$\text{tangent} = \rho\sqrt{1+\rho^2\frac{d\theta^2}{d\rho^2}} = \rho\sqrt{1+\frac{\rho^2}{a^2}};$$

$$\text{normal} = \sqrt{\rho^2+a^2}; \quad p = \frac{\rho^2}{\sqrt{\rho^2+a^2}}.$$

2. Find the subtangent, subnormal, tangent, and normal of the logarithmic spiral $\rho = a^\theta$.

Ans. $\text{subt.} = \frac{\rho}{\log a}$; $\quad \text{subn.} = \rho \log a$;

$$\tan. = \rho\sqrt{1+\frac{1}{(\log a)^2}}; \quad \text{nor.} = \rho\sqrt{1+(\log a)^2}.$$

Since $\tan \text{OPT} = \frac{\rho\, d\theta}{d\rho} = \frac{1}{\log a}$, this curve makes the same angle with every radius vector, and therefore is called the *equiangular spiral.*

If $a = e$, $\tan \text{OPT} = 1$, $\text{OPT} = \frac{\pi}{4}$, subtangent = subnormal, and tangent = normal.

3. Find the subtangent, subnormal, and p of the lemniscate of Bernouilli $\rho^2 = a^2 \cos 2\theta$.

Ans. $\text{subt.} = \frac{-\rho^3}{a^2 \sin 2\theta}$; $\quad \text{subn.} = \frac{-a^2}{\rho}\sin 2\theta$;

$$p = \frac{\rho^3}{\sqrt{\rho^4 + a^4 \sin^2 2\theta}} = \frac{\rho^3}{a^2}.$$

Rectilinear Asymptotes.

138. A Rectilinear Asymptote is a straight line that has the limiting position of the tangent to an infinite branch of a curve. If a curve has no infinite branch, it evidently can have no asymptote.

If X and Y represent the intercepts of a tangent on the axes of x and y respectively, from the equation of the tangent to any plane curve

$$y - y' = \frac{dy'}{dx'}(x - x'),$$

we obtain $X = x' - y'\dfrac{dx'}{dy'}$, (1)

and $Y = y' - x'\dfrac{dy'}{dx'}$. (2)

Now if, as the point of contact (x', y') moves out along an infinite branch, the value of X or Y or the values of both approach finite limits, it is evident that these limits will be the intercepts on the axes of an asymptote to that branch of the curve.

EXAMPLES.

1. Examine $y^3 = 6x^2 + x^3$ for asymptotes.

Solving for y, we have

$$y = x\sqrt[3]{\frac{6}{x} + 1}.$$

As $x \doteq \infty$,* $y \doteq \infty$; and, as $x \doteq -\infty$, $y \doteq -\infty$.

Hence the curve has two infinite branches, one in the first angle and another in the third.

$$X = x' - \frac{y'^3}{4x' + x'^2} = -\frac{2}{\dfrac{4}{x'} + 1} \doteq -2 \text{ as } x' \doteq \pm\infty;$$

$$Y = y' - \frac{4x'^2 + x'^3}{y'^2} = \frac{2}{\left(\dfrac{6}{x'} + 1\right)^{\frac{2}{3}}} \doteq 2 \text{ as } x' \doteq \pm\infty.$$

---

* $x \doteq a$ as $y \doteq \infty$ is read "x approaches a as its limit as y approaches infinity, or increases without limit." A variable cannot approach infinity *as its limit*. For example, if $y = \dfrac{1}{x}$, and $x \doteq 0$, y does not approach infinity *as its limit*; for, when x is infinitely near 0, y is infinitely large; but it doubles its value while x decreases by half its own value. Hence, as $x \doteq 0$, the difference between ∞ and y must always be many times as great as y, however great y may become.

Therefore, the line $y = x + 2$, whose intercepts on the axes of x and y are respectively -2 and 2, is an asymptote to each branch.

2. Examine the conic sections for asymptotes.

Neither the circle nor the ellipse can have an asymptote, since neither has an infinite branch.

The parabola has two infinite branches, one in the first angle and another in the fourth.

Here $$X = x' - \frac{y'^2}{p} = -x' \doteq -\infty \text{ as } x' \doteq \infty;$$

and $$Y = y' - \frac{px'}{y'} = \frac{y'}{2} \doteq \infty \text{ as } x' \doteq \infty.$$

Hence the parabola has no asymptote.

The hyperbola has four infinite branches, one in each angle. In this curve

$$X = x' - \frac{a^2y'^2}{b^2x'} = \frac{a^2}{x'} \doteq \pm 0 \text{ as } x' \doteq \pm\infty;$$

$$Y = y' - \frac{b^2x'^2}{a^2y'} = -\frac{b^2}{y'} \doteq \pm 0 \text{ as } x' \text{ or } y' \doteq \pm\infty.$$

Hence the asymptote to each branch passes through the origin. To determine the direction of these asymptotes, we have

$$\frac{dy'}{dx'} = \frac{b^2x'}{a^2y'} = \pm\frac{b}{a\sqrt{1 - \frac{a^2}{x'^2}}} \doteq \pm\frac{b}{a} \text{ as } x' \doteq \pm\infty.$$

Since $\frac{b^2x'}{a^2y'}$ is positive for any point in the first or third angle, and negative for any point in the second or fourth, $y = \frac{b}{a}x$ is the asymptote to the branches in the first and the third angle, and $y = -\frac{b}{a}x$ to those in the second and the fourth. These asymptotes are evidently the produced diagonals of the rectangle on the axes.

3. Prove that $y = -x$ is an asymptote to each of the two infinite branches of $y^3 = a^3 - x^3$.

4. Show that $y^2 = ax^3$ has no asymptotes.

139. Asymptotes Determined by Inspection or Expansion. From the definition of an asymptote, it follows that it is a line which an infinite branch of a curve approaches indefinitely near, but never reaches. From this view of asymptotes, we can often determine their equations by *inspecting* the equation of the curve, or *expanding* one of its members. Thus, in the cissoid

$$y^2 = \frac{x^3}{2a - x},\ y \doteq \pm \infty \text{ as } x \doteq 2a.$$

Whence, $x = 2a$ is an asymptote to the two infinite branches of the curve; for they approach indefinitely near, but never reach, this line.

In $x = \log_a y$, or $y = a^x$, $x \doteq -\infty$ as $y \doteq 0$. The axis of x is therefore an asymptote to the infinite branch in the second angle.

Again, the equation $xy - ay - bx = 0$ may be put in the form,

$$y = \frac{bx}{x - a} \text{ or } x = \frac{ay}{y - b},$$

from which we know that $x = a$ and $y = b$ each is an asymptote to two infinite branches.

The method of examining a curve for asymptotes by solving its equation for y, and then developing the second member in descending powers of x, by Maclaurin's formula or some other means, will be illustrated by a few examples.*

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* The following is a brief view of another method of examining a curve for asymptotes. For a fuller treatment, see Williamson's *Differential Calculus*, page 240.

Let the equation of a right line be

$$y = \mu x + \nu; \tag{1}$$

that of a curve of the nth degree,

$$f(x, y) = 0; \tag{2}$$

that obtained by substituting $\mu x + \nu$ for y in (2),

$$\phi(x) = 0. \tag{3}$$

EXAMPLES.

1. Examine $x^3 - xy^2 + ay^2 = 0$ for asymptotes.

Here $$y = \pm x\left(\frac{x}{x-a}\right)^{\frac12} = \pm x\left(1-\frac{a}{x}\right)^{-\frac12}$$
$$= \pm x\left(1+\frac{a}{2x}+\frac{3a^2}{8x^2}+\cdots\right).$$

From the first form of the value of y, $x = a$ is evidently an asymptote to two branches of the curve that lie to the right of it. From the last form we see that two branches of the curve approach infinitely near each of the lines $y = \pm x \pm \frac{a}{2}$, as $x \doteq \pm \infty$. The curve therefore has three asymptotes, each of which is asymptotic to two infinite branches.

Equation (3) is evidently of the nth degree, and its n roots are the abscissas of the n real or imaginary intersections of (1) and (2). If two roots of (3) be equal, two points of intersection of (1) and (2) will coincide, and, in general, (1) will be a tangent to (2). From Algebra, we know that, as the coefficients of x^n and x^{n-1} approach zero as a limit, two roots of (3) increase without limit. Hence, if μ and ν in (1) have such values as render these coefficients 0, (1) has the limiting position of a tangent to an infinite branch, or is an asymptote.

For example, let the curve be

$$y^3 = ax^2 + x^3. \tag{1}$$

Substituting $\mu x + \nu$ for y, and arranging the terms, we have

$$(\mu^3 - 1)x^3 + (3\mu^2\nu - a)x^2 + 3\mu\nu^2 x + \nu^3 = 0. \tag{2}$$

Two roots of (2) become ∞, when $\mu^3 - 1 = 0$, and $3\mu^2\nu - a = 0$; that is, when $\mu = 1$, and $\nu = \frac13 a$. Hence, $y = x + \frac13 a$ is an asymptote to (1).

From the theory of equations, and this theory of asymptotes, the following are obvious conclusions:

(*a*) Any asymptote or tangent to a curve of the third degree intersects the curve in one, and only one, point.

(*b*) Any asymptote or tangent to a curve of the nth degree cannot meet it in more than $n - 2$ points, exclusive of the point of contact.

2. Examine $y^2 = x^2\dfrac{x^2-1}{x^2+1}$ for asymptotes.

$$y = \pm x\left(1-\frac{1}{x^2}\right)^{\frac{1}{2}}\left(1+\frac{1}{x^2}\right)^{-\frac{1}{2}}$$
$$= \pm x\left(1-\frac{1}{2x^2}-\cdots\right)\left(1-\frac{1}{2x^2}+\cdots\right)$$
$$= \pm x\left(1-\frac{1}{x^2}+\cdots\right).$$

Hence $y = \pm x$ are the two asymptotes.

3. Examine $y^3 = ax^2 - x^3$ for asymptotes.

Ans. $y = -x + \dfrac{a}{3}$.

4. Examine $y = c + \dfrac{a^3}{(x-b)^2}$ for asymptotes.

Ans. $y = c$, and $x = b$.

5. Examine $y^2 = \dfrac{x^3+ax^2}{x-a}$ for asymptotes.

Ans. $x = a$, and $y = \pm(x+a)$.

140. Asymptotes to Polar Curves. If, as $\theta \doteq \theta'$, $\rho \doteq \infty$, and the subtangent OD $\doteq$ OS, it is evident that SN, which is parallel to OM, is an asymptote to the infinite branch PK. Hence, to examine a polar curve for asymptotes, we find from its equation the values of θ which make $\rho = \pm\infty$. If the corresponding value of the subtangent is finite, the line parallel to the infinite radius vector, and passing through the extremity of the limiting subtangent, is an asymptote.

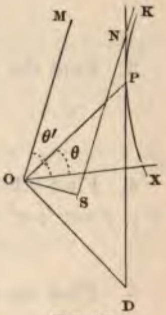

Fig. 37.

EXAMPLES.

1. Examine the hyperbolic spiral $\rho\theta = a$ for asymptotes.

Here $\rho = \dfrac{a}{\theta}$; hence, when $\theta = 0$, $\rho = \infty$, and subtangent $= -a$. The curve therefore has an asymptote parallel to the initial line, and at the distance a above it.

2. Examine $\rho \cos\theta = a \cos 2\theta$ for asymptotes.

Here $\rho = \frac{a \cos 2\theta}{\cos\theta}$; hence, when $\theta = \frac{\pi}{2}$, $\rho = \pm\infty$, and subtangent $= -a$.

The line perpendicular to the initial line at the distance a to the left of the pole is therefore an asymptote to two infinite branches.

3. Examine $\rho^2 \cos\theta = a^2 \sin 3\theta$ for asymptotes.

Ans. The perpendicular to the initial line at the origin is an asymptote.

4. Show that the initial line is an asymptote to the lituus $\rho\sqrt{\theta} = a$.

Miscellaneous Examples.

1. At what angle does $y^2 = 10x$ intersect $x^2 + y^2 = 144$?

Ans. 71° 0′ 58″.

2. Find the subnormal of the curve $y^2 = 2a^2 \log x$.

Ans. $\frac{a^2}{x}$.

3. Find the equation of the tangent to the curve

$$x^2(x+y) = a^2(x-y) \text{ at the origin.}$$

Ans. $y = x$.

4. Find at what angle the curve $y^2 = 2ax$ cuts the curve $x^3 - 3axy + y^3 = 0$.

Ans. $\cot^{-1}\sqrt[3]{4}$.

5. Find the normal, subnormal, tangent, and subtangent of the catenary $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$.

Ans. Subn. $= \frac{a}{4}(e^{\frac{2x}{a}} - e^{-\frac{2x}{a}})$; norm. $= \frac{y^2}{a}$.

subt. $= \frac{ay}{\sqrt{y^2 - a^2}}$; tan. $= \frac{y^2}{\sqrt{y^2 - a^2}}$.

6. Find the subtangent of the curve $y^2 = \dfrac{x^3}{a - x}$.

Ans. $\dfrac{2x(a - x)}{3a - 2x}$.

7. Examine $y^2(x - 2a) = x^3 - a^3$ for asymptotes.

Ans. $x = 2a$; $y = \pm(x + a)$.

8. Examine $y(a^2 - x^2) = b^2(2x + e)$ for asymptotes.

Ans. $y = 0$; $x + a = 0$; $x = a$.

9. Examine for asymptotes the folium of Descartes,
$x^3 + y^3 - 3axy = 0$.

Ans. $y = -x - a$.

10. Examine $(y^2 - 1)y = (x^2 - 4)x$ for asymptotes.

Ans. $y = x$.

11. Examine $y^2 = \dfrac{a^2(x - a)(x - 3a)}{x^2 - 2ax}$ for asymptotes.

Ans. $x = 2a$; $x = 0$; $y = \pm a$.

12. Examine the hyperbola $\rho = \dfrac{a(e^2 - 1)}{e \cos\theta - 1}$ for asymptotes.

13. Find the length of the perpendicular from the pole upon the tangent to the lituus $\rho\sqrt{\theta} = a$.

Ans. $p = \dfrac{2a^2\rho}{\sqrt{\rho^4 + 4a^4}}$.

14. In the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, prove that the portion of the tangent intercepted between the axes equals a.

15. Give the different methods of drawing a tangent to any plane curve at a given point.

CHAPTER XI.

DIRECTION OF CURVATURE, SINGULAR POINTS, AND CURVE TRACING.

141. Direction of Curvature. A curve is concave upward or downward at any point, according as in the immediate vicinity of that point it lies above or below the tangent at that point.

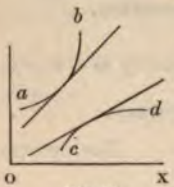

Fig. 38.

When a curve, as *ab*, is concave upward, it is evident that $\frac{dy}{dx}$, the slope of the curve, increases as x increases; hence, $\frac{d^2y}{dx^2}$, the derivative of $\frac{dy}{dx}$, is positive (§ 72). When a curve, as *cd*, is concave downward, $\frac{dy}{dx}$ decreases as x increases, and $\frac{d^2y}{dx^2}$ is negative.

Hence, *the curve* $y = f(x)$ *is concave upward or downward, at any point* (x, y), *according as* $\frac{d^2y}{dx^2}$ *is positive or negative.*

In the *polar* system, a curve is said to be concave or convex toward the pole at any point according as in the immediate vicinity of that point it lies on the same side of the tangent as the pole, or on the opposite side. From the figure, it is evident that, when the curve is concave toward the pole, p or OD increases as ρ increases, and $\frac{dp}{d\rho}$ is positive (§ 31). When the curve is convex toward the pole, p decreases as ρ increases, and $\frac{dp}{d\rho}$ is negative.

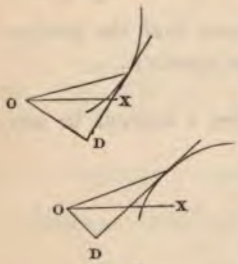

Fig. 39.

Hence, *a polar curve is concave or convex toward the pole according as* $\frac{dp}{d\rho}$ *is positive or negative.*

When the equation of the curve is given in terms of ρ and θ, we find p in terms of ρ by use of (8) of § 136.

EXAMPLES.

1. Find the direction of curvature of $y = x^2 + 2x + 5$.

Here $\frac{d^2y}{dx^2} = 2$; hence the curve is concave upward.

2. Find the direction of curvature of $y = a + c(x + b)^3$.

Here $\frac{d^2y}{dx^2} = 6c(x + b)$; hence the curve is concave upward or downward at (x, y), according as $x >$ or $< -b$.

3. Find the direction of curvature of $y = x^3 - 3x^2 - 9x + 9$.

Ans. Concave upward or downward according as $x >$ or < 1.

4. Find the direction of curvature of $x = \log_a y$, and $y = \sin x$.

5. Find the direction of curvature of the lituus $\rho^2\theta = a^2$.

Here
$$\frac{d\rho}{d\theta} = -\frac{\rho^2}{2\rho\theta} = -\frac{\rho^3}{2a^2};$$

$$\therefore p = \frac{\rho^2}{\sqrt{\rho^2 + \frac{d\rho^2}{d\theta^2}}} = \frac{2a^2\rho}{\sqrt{4a^4 + \rho^4}}; \qquad \text{§ 136,(8)}$$

$$\therefore \frac{dp}{d\rho} = \frac{2a^2(4a^4 - \rho^4)}{(4a^4 + \rho^4)^{\frac{3}{2}}}.$$

Hence this spiral is concave or convex toward the pole at (θ, ρ) according as $\rho <$ or $> a\sqrt{2}$.

6. Find the direction of curvature of the logarithmic spiral $\rho = a^\theta$.

Here
$$p = \frac{\rho}{\sqrt{1 + (\log a)^2}}; \quad \therefore \frac{dp}{d\rho} = \frac{1}{\sqrt{1 + (\log a)^2}}.$$

Hence the curve is concave toward the pole.

Singular Points.

142. **Singular Points** of a curve are those which have some *peculiar* property. Such points are: first, **Points of Inflexion**; second, **Multiple Points**; third, **Conjugate Points**; fourth, **Stop Points.**

143. **Points of Inflexion.** A *point of inflexion* is a point at which a curve changes its direction of curvature. Hence a tangent at a point of inflexion intersects the curve. Thus the tangent at P, a point of inflexion, cuts the curve at P.

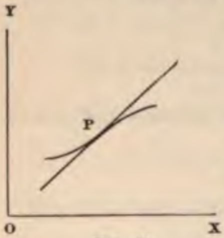

FIG. 40.

At a point of inflexion on $y=f(x)$, $\frac{d^2y}{dx^2}$ must evidently change its sign, and therefore pass through 0 or ∞. Hence, if $\frac{d^2y}{dx^2}$ be found in terms of x, the roots of $\frac{d^2y}{dx^2}=0$ or ∞ are the *critical* values of x to be examined. If $\frac{d^2y}{dx^2}$ changes its sign as x passes through any one of these values, this value is the abscissa of a point of inflexion.*

EXAMPLES.

1. Examine $x^3-3bx^2+a^2y=0$ for points of inflexion.

The root of $\frac{d^2y}{dx^2}=\frac{6(b-x)}{a^2}=0$ is b; and $\frac{6(b-x)}{a^2}$ evidently changes its sign as x passes through b; hence $\left(b, \frac{2b^3}{a^2}\right)$ is a point of inflexion.

2. Prove that the points in which $y=c\sin\frac{x}{a}$ cuts the axis of x are all points of inflexion.

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* On one side of a point of inflexion, the slope of a curve is increasing; and, on the other, it is decreasing: hence a point of inflexion is a point of maximum or minimum slope, and the method of finding such a point is seen to be that of finding a maximum or minimum of $\frac{dy}{dx}$.

The roots of $\frac{d^2y}{dx^2} = -\frac{c}{a^2}\sin\frac{x}{a} = 0$ are $0, a\pi, 2a\pi, 3a\pi$, etc. As x passes through each of these values, $\frac{d^2y}{dx^2}$ changes its sign; hence $0, a\pi, 2a\pi, 3a\pi$, etc., are abscissas of points of inflexion.

3. Examine the witch of Agnesi, $x^2y = 4a^2(2a - y)$, for points of inflexion.

Ans. $(\pm\frac{2}{3}a\sqrt{3}, \frac{3}{2}a)$.

4. Examine $y = \frac{x^3}{a^2 + x^2}$ for points of inflexion.

Ans. $(0, 0)$, $(a\sqrt{3}, \frac{3}{4}a\sqrt{3})$, and $(-a\sqrt{3}, -\frac{3}{4}a\sqrt{3})$.

144. To test *curves* given by their *polar equations* for points of inflexion, we find the roots of $\frac{dp}{d\rho} = 0$ or ∞. If $\frac{dp}{d\rho}$ changes its sign as ρ passes through any one of these critical values, this value is the radius vector of a point of inflexion.* Thus, in the lituus, $(28° 38', a\sqrt{2})$ is a point of inflexion; for $\frac{dp}{d\rho}$ changes its sign as ρ passes through $a\sqrt{2}$, the root of $\frac{dp}{d\rho} = 0$. (See § 141, Ex. 5.)

Multiple Points.

145. A **Multiple Point** is one through which two or more branches of a curve pass, or at which they meet. A multiple point is double when there are only two branches, triple when only three, and so on.

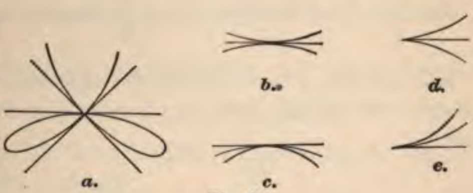

Fig. 41.

A multiple point at which the branches intersect (Fig. *a*) is called a **Multiple Point of Intersection.**

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* A point of inflexion on a polar curve evidently corresponds to a maximum or minimum of p.

A multiple point through which two branches pass, and at which they are tangent (Figs. *b*, *c*) is an **Osculating Point.**

A multiple point at which two branches terminate, and are tangent (Figs. *d*, *e*) is a **Cusp.** A cusp or osculating point is said to be of the first or the second species, according as the two branches are on opposite sides (Figs. *b*, *d*) or the same side (Figs. *c*, *e*) of their common tangent.

A **Conjugate Point** is one that is entirely isolated from the rest of the real locus. Hence, in an algebraic curve, a conjugate point is a multiple point formed by the intersection or meeting, in the plane of the axes, of imaginary branches; that is, of branches lying outside of the plane of the axes. Since an odd number of roots of an algebraic equation cannot be imaginary, an even number of imaginary branches must intersect or touch in a conjugate point of an algebraic curve.

146. From the definitions given above, it follows that, at a multiple point of intersection, $\frac{dy}{dx}$ must have two or more unequal real values; that, at a point of osculation or a cusp it must have two equal real values; that, at a conjugate point on an algebraic curve, it must have two or more values which are imaginary, unless the tangents to the imaginary branches at the conjugate point lie in the plane of the axes.

Hence *at any multiple point* $\frac{dy}{dx}$ *has two or more values.*

147. *If* $f(x, y) = u = 0$ *be the algebraic equation of a curve freed from radicals and fractions, at any multiple point upon the curve,*

$$\frac{dy}{dx} = -\frac{\frac{du}{dx}}{\frac{du}{dy}} = \frac{0}{0}, \text{ or } \frac{du}{dx} = \frac{du}{dy} = 0.$$

For, at any multiple point, $\frac{dy}{dx}$, or the ratio of $\frac{du}{dx}$ to $\frac{du}{dy}$, must have two or more values (§ 146). But, from the form of the equation of the curve, neither $\frac{du}{dx}$ nor $\frac{du}{dy}$ can contain radicals or fractions; hence their ratio can have two or more values for the same values of x and y, only when it assumes the indeterminate form $\frac{0}{0}$.

148. Examination of a Curve for Multiple Points. To examine a curve for multiple points, put its equation in the required form $f(x, y) = u = 0$, and find the sets of values of x and y that will satisfy the equations $\frac{du}{dx} = 0$ and $\frac{du}{dy} = 0$. Of these sets, those which satisfy the equation of the curve give the points to be examined.

Let (x', y') be one of these points; then

$$\frac{dy'}{dx'} = -\frac{\frac{du}{dx'}}{\frac{du}{dy'}} = \frac{0}{0},$$

which is evaluated according to the method of § 83.

I. If $\frac{dy'}{dx'}$ has two or more unequal real values, (x', y') is in general a *multiple point of intersection*.

II. If $\frac{dy'}{dx'}$ has two equal real values, (x', y') is in general either an *osculating point* or a *cusp*.

III. If all the values of $\frac{dy'}{dx'}$ are imaginary, (x', y') is a *conjugate point*.

The following considerations enable us to discover more exactly the nature of these points:

In Case I., if the values of y or $\frac{dy}{dx}$ are real for $x = x' - h$ and $x = x' + h$, h being very small, (x', y') is a multiple point of intersection; if real for neither, (x', y') is a conjugate point at which the imaginary branches are parallel to the plane of the axes.

In Case II., if the values of y or $\frac{dy}{dx}$ are real for $x = x' - h$ and $x = x' + h$, (x', y') is a point of osculation; if real for only one of these values of x, (x', y') is a cusp; if real for neither, (x', y') is a conjugate point.

In some curves, and especially when $\frac{dy'}{dx'} = \infty$, it is better to inspect the values of x or $\frac{dx}{dy}$ for $y = y' - h$ and $y = y' + h$.

To determine the species of a cusp or point of osculation, find $\frac{d^2y}{dx^2}$, and (x', y') will be of the first or the second species, according as the two values of $\frac{d^2y'}{dx'^2}$ have opposite signs or the same sign. Or we may compare the ordinates or abscissas of adjacent corresponding points on the branches and on their common tangent.*

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* The following is a brief view of another method of examining a curve for multiple points. For a fuller treatment, see Williamson's *Differential Calculus* or Salmon's *Higher Plane Curves*.

Let the equation of a curve of the nth degree be

$$f(x, y) = 0; \tag{1}$$

that of a right line through the origin,

$$y = \mu x; \tag{2}$$

that obtained by substituting μx for y in (1),

$$\phi(x) = 0. \tag{3}$$

If (1) contains no constant term, its locus evidently passes through the origin. If (1) contains no constant term nor any term of the first degree, two roots of (3) are 0. Hence two points of intersection of (1) and (2) are at the origin; and the origin is a double point. If, in addition, (1) contains no term of the second degree, three roots of (3) are 0, and the origin is a triple point.

Hence, when the origin is a multiple point, this fact is evident from the equation of the curve. To examine a curve for multiple points not at the origin, change the reference of the locus to new parallel axes, using the formulas $x = m + x_1$, and $y = n + y_1$. If m and n can be so determined that the resulting equation will contain no constant term nor any term of the first degree in x_1 or y_1, (m, n), or the new origin, is a double point. If m and n can be so determined that the new equation will contain no constant

EXAMPLES.

1. Examine $x^4+ax^2y-ay^3=0$ for multiple points.

Here $$u=x^4+ax^2y-ay^3=0; \tag{1}$$

$$\therefore \frac{du}{dx}=4x^3+2axy, \text{ and } \frac{du}{dy}=ax^2-3ay^2;$$

$$\therefore \frac{dy}{dx}=\frac{4x^3+2axy}{3ay^2-ax^2}. \tag{2}$$

Placing the partial derivatives equal to zero, we have

$$x(2x^2+ay)=0, \tag{3}$$

and $$x^2-3y^2=0. \tag{4}$$

Solving (3) and (4), we obtain the following three sets of values for x and y,

$$x=0, \qquad \tfrac{1}{6}a\sqrt{3}, \qquad \text{and } -\tfrac{1}{6}a\sqrt{3},$$
$$y=0, \qquad -\tfrac{1}{6}a, \qquad \text{and } -\tfrac{1}{6}a.$$

Only the first set of values will satisfy (1); and (0, 0) is the only point to be examined. From (2) we have

$$\left.\frac{dy}{dx}\right]_{0,0}=\left.\frac{4x^3+2axy}{3ay^2-ax^2}\right]_{0,0}=0 \text{ and } \pm 1. \qquad \S\ 83.$$

Hence the origin is a triple point at which the inclinations of the branches are respectively 0, $\frac{1}{4}\pi$, and $\frac{3}{4}\pi$.

From (1),

$$x=\pm\sqrt{-\tfrac{1}{2}ay\pm\tfrac{1}{2}y\sqrt{4ay+a^2}}. \tag{5}$$

From (5), we see that four values of x are real when $y=-h$, and two when $y=+h$; hence each of the three branches passes

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term nor any of the first or second degree in x_1 or y_1, (m, n) is a triple point.

From this method, it is evident that a curve of the third degree can have only one multiple point; one of the fourth degree, only two double points or one triple point; one of the fifth degree, only two double points or one triple and one double point.

through the origin, which is therefore a triple point of intersection. The general form of the curve at the origin is shown in Fig. a on page 157.

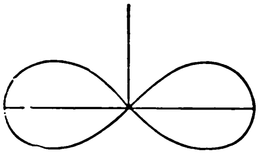

Fig. 42.

2. Examine $y^2 = x^2(a^2 - x^2)$ for multiple points.

Ans. (0, 0), Fig. 42, is a double point of intersection; $\left.\frac{dy}{dx}\right]_{0,0} = \pm a$.

3. Examine $x^3 - 3axy + y^3 = 0$ for multiple points.

Ans. (0, 0) is a double point of intersection; $\left.\frac{dy}{dx}\right]_{0,0} = 0$ and ∞.

4. Examine $y^2(a^2 - x^2) = x^4$ for multiple points.

$$\left.\frac{dy}{dx}\right]_{0,0} = \pm 0,$$

and from its equation we see that the curve extends through the origin, and is symmetrical with respect to the axis of x; hence the origin is a point of osculation of the first species.

5. Examine $y^2 = a^2x^3$ for multiple points.

$$\left.\frac{dy}{dx}\right]_{0,0} = \pm 0,$$

and from its equation we see that the curve consists of two branches symmetrical with respect to the axis of x, and extending from the origin to the right; hence the origin is a cusp of the first species, the common tangent being the axis of x (Fig. 43).

6. Examine $y^3 = ax^2 - x^3$ for multiple points.

Here (0, 0) is the only critical point. $\left.\frac{dy}{dx}\right]_{0,0} = \pm\infty$, and $y = (ax^2 - x^3)^{\frac{1}{3}}$ shows that there is a branch on each side of the axis of y, neither of which extends below the origin, which is therefore a cusp of the first species.

7. Examine $y^2 = x(x + a)^2$ for multiple points.

Here $(-a, 0)$ is the critical point, and $\left.\frac{dy}{dx}\right]_{-a,0} = \pm\sqrt{-a}$; hence $(-a, 0)$ is a conjugate point.

8. **Examine** $a^3y^2 - 2abx^2y - x^5 = 0$ **for singular points.**

Here $(0, 0)$ is the critical point, and $\left.\frac{dy}{dx}\right]_{0,0} = \pm 0$.

When $x = 0 + h$,

$$y = \frac{bh^2}{a^2} \pm \sqrt{\frac{b^2h^4 + ah^5}{a^4}},$$

in which both the values of y are real, one greater and the other less than 0.

When $x = 0 - h$,

$$y = \frac{bh^2}{a^2} \pm \sqrt{\frac{b^2h^4 - ah^5}{a^4}},$$

in which both the values of y are real and greater than 0 when h is small.

Hence the origin is a point of osculation, and a point of inflexion on one branch.

149. A multiple point at which two or more branches terminate, and have different tangents, is a **Shooting Point.** A **Stop Point** is a point at which a single branch of a curve terminates.

From the law of imaginary roots of algebraic equations, neither a shooting point nor a stop point can occur on an algebraic curve.

150. Curve Tracing. The most rudimentary method of tracing a curve is to find from its equation such a number of its points that, when located, these points will clearly indicate the form of the curve.

This method is laborious; and our present object is to utilize the principles heretofore developed to determine directly from its equation the general form of a curve, especially at such points as present any peculiarity, so that the curve may be traced without the labor of the first method.

To *trace* a curve from its *rectilinear equation*, the following general directions will be found useful.

Solve its equation for y or x, and determine any lines or points with respect to which the curve is symmetrical.

Find the points at which the curve cuts the axes, and determine its limits and infinite branches.

Determine the positions of its asymptotes, and on which side of each the infinite branches lie.

Find its maxima and minima ordinates, and the angles at which it cuts the axes.

Determine its direction of curvature, points of inflexion, and multiple points.

EXAMPLES.

1. Trace the curve whose equation is $y^2 = a^2x^3$.

Here $y = \pm ax^{\frac{3}{2}}$, and the curve is symmetrical with respect to the axis of x.

When $x = 0$, $y = 0$, and the curve meets the axes at $(0, 0)$.

When $x < 0$, y is imaginary; but when $x > 0$, y is real. Hence there is one infinite branch in the first angle, and another in the fourth.

$$\frac{dy}{dx} = \frac{3a^2x^2}{2y} = \pm\frac{3ax^{\frac{1}{2}}}{2} \doteq \infty \text{ when } x \doteq \infty;$$

hence the curve has no asymptote.

$$\frac{dy}{dx} = \frac{3a^2x^2}{2y} = \frac{0}{0} \text{ at } (0, 0);$$

$$\text{but } \left.\frac{dy}{dx}\right]_{0,0} = \pm\left.\frac{3ax^{\frac{1}{2}}}{2}\right]_{0,0} = \pm 0.$$

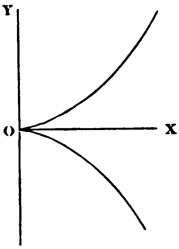

Fig. 43.

Hence the two symmetrical branches terminating at the origin are tangent to the axis of x at that point, and the origin is a cusp of the first species.

Since $\frac{d^2y}{dx^2} = \pm\frac{3a}{4\sqrt{x}}$, the upper branch is concave upward, and the lower one concave downward. The form of the curve is shown in Fig. 43.

2. Trace the curve $y^3 = 2ax^2 - x^3$.

Here $y = (2ax^2 - x^3)^{\frac{1}{3}}$.

For $y = 0$, $x = 0$ and $2a$; hence the curve cuts the axis of x at the origin, and $2a$ at the right of it.

For each real value of x, y has one, and only one, real value, which is + or − according as $x <$ or $> 2a$. Hence there is one infinite branch in the second angle and another in the fourth.

To find the equation of the asymptote, we have

$$y = (2ax^2 - x^3)^{\frac{1}{3}} = -x\left(1 - \frac{2a}{x}\right)^{\frac{1}{3}}$$

$$= -x\left(1 - \frac{2a}{3x} - \frac{4a^2}{9x^2} - \cdots\right), \qquad (1)$$

when x is numerically greater than $2a$.

Hence the equation of the asymptote to each infinite branch is

$$y = -x + \tfrac{2}{3}a. \qquad (2)$$

From (1) and (2), it is evident that the infinite branches lie between the asymptote and the axis of x.

$$\frac{dy}{dx} = \frac{4ax - 3x^2}{3y^2} = 0 \text{ when } x = \tfrac{4}{3}a;$$

$$\therefore (2ax^2 - x^3)^{\frac{1}{3}}\Big]_{\frac{4}{3}a}, \text{ or } \tfrac{2}{3}a\sqrt[3]{4},$$

is a maximum ordinate.

$$\frac{dy}{dx}\Big]_{2a,0} = \infty;$$

hence the curve is perpendicular to the axis of x at $(2a, 0)$.

$$\frac{dy}{dx} = \frac{4ax - 3x^2}{3y^2} = \frac{0}{0}, \text{ when } x = y = 0.$$

By evaluating we find that

$$\frac{dy}{dx}\Big]_{0,0} = \pm\infty;$$

hence, as the curve does not extend below the axis of x to the left of $x = 2a$, and y has one, and but one, real value for each value of x, the origin is a cusp of the first species, the two branches being tangent to the axis of y.

$$\frac{d^2y}{dx^2}=\frac{-8a^2}{9x^{\frac{4}{3}}(2a-x)^{\frac{5}{3}}},$$

which is + or −, according as $x >$ or $< 2a$; hence $(2a, 0)$ is a point of inflexion, to the right of which the curve is concave upward, and to the left downward. The form of the curve is given in Fig. 44.

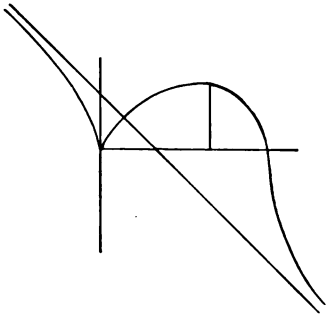

Fig. 44.

3. Trace the curve

$$y^2(x^2-a^2)=x^4.$$

Since its equation involves only even powers of x and y, the curve is symmetrical with respect to each axis. Hence, if we determine the part of the locus that is in the first angle, the symmetry of the curve will give us the other three parts.

Solving for y, we have

$$y=\pm\frac{x^2}{(x^2-a^2)^{\frac{1}{2}}}.$$

When $x=0$, $y=0$; but, for other values of x between $-a$ and $+a$, y is imaginary; hence $(0, 0)$ is a conjugate point, and the locus in the first angle lies to the right of $x=a$.

As $x \doteq a$ from a value greater than a, $y \doteq \infty$; hence there is one infinite branch in the first angle to which $x=a$ is the asymptote. When $x \doteq \infty$, $y \doteq \infty$; and there is a second infinite branch in the first angle.

To find the equation of the other asymptote, we have

$$y=\pm\frac{x^2}{x\left(1-\frac{a^2}{x^2}\right)^{\frac{1}{2}}}=\pm x\left(1-\frac{a^2}{x^2}\right)^{-\frac{1}{2}}$$

$$=\pm x\left(1+\frac{a^2}{2x^2}+\cdots\right),$$

when x is numerically greater than a; hence $y=x$ is the equation of the other asymptote.

Evidently the curve lies above this asymptote. Hence the branch in the first angle lies above $y = x$, and to the right of $x = a$.

$$\frac{dy}{dx} = \frac{2x^3 - xy^2}{y(x^2 - a^2)} = 0,$$

when $y^2 = 2x^2$, or $x = \pm a\sqrt{2}$; hence $2a$ is a minimum ordinate.

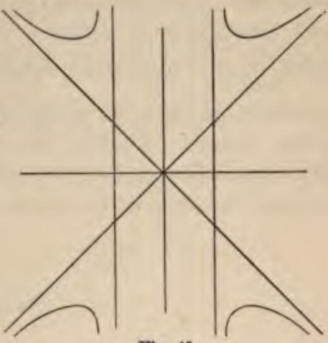

Fig. 45.

In the first angle,

$$\frac{d^2y}{dx^2} = \frac{a^2(x^2 + 2a^2)}{(x^2 - a^2)^{\frac{5}{2}}}.$$

As this is + when $x > a$, this branch in the first angle is concave upward. The form of the curve is given in Fig. 45.

4. Trace the curve $\dfrac{x^3}{a^3} + \dfrac{y^3}{b^3} = 1.$

The curve cuts the axis of x at $(a, 0)$ and the axis of y at $(0, b)$. There is one infinite branch in the second angle and another in the fourth. $y = -\dfrac{b}{a}x$ is the equation of the asymptote, which lies below the infinite branches. The curve is concave upward, except between $x = 0$ and $x = a$, where it is concave downward. $(0, b)$ and $(a, 0)$ are points of inflexion.

5. Trace the curve $y = \dfrac{x}{1 + x^2}$.

The curve has one infinite branch in the first angle and another in the third, to each of which the axis of x is an asymptote. $\frac{1}{2}$ is a maximum, and $-\frac{1}{2}$ a minimum, ordinate. $(0, 0)$, $(-\sqrt{3}, -\frac{1}{4}\sqrt{3})$, and $(\sqrt{3}, \frac{1}{4}\sqrt{3})$ are points of inflexion. The inclination of the curve at the origin is $\frac{1}{4}\pi$.

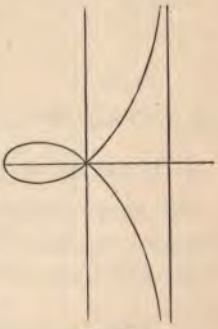

Fig. 46.

6. Trace the curve $y^2 = x^2\dfrac{a + x}{a - x}$. For the form of the curve, see Fig. 46.

151. Tracing Polar Curves. When possible, write the equation in the form $\rho = f(\theta)$. Solve $f(\theta) = 0$ to find the angles at which the curve cuts the polar axis at the pole. Assign to θ such positive and negative values as make ρ easily found. Solve $\frac{d\rho}{d\theta} = 0$ to find the values of θ for which ρ is a maximum or minimum, and for which the curve is perpendicular to the radius vector. Examine the curve for asymptotes, direction of curvature, and points of inflexion. The facts thus obtained will indicate the form of the curve.

EXAMPLES.

1. Trace the curve $\rho = a \sin 3\theta$.

Since $\rho = a \sin 3\theta$, ρ reaches its maximum value a when $\sin 3\theta = 1$; that is, when $\theta = \frac{1}{6}\pi, \frac{5}{6}\pi, \frac{3}{2}\pi$, etc.; and ρ reaches its minimum $-a$ when $\sin 3\theta = -1$; that is, when $\theta = \frac{1}{2}\pi, \frac{7}{6}\pi, \frac{11}{6}\pi$, etc.

Since $\frac{d\rho}{d\theta} = 3a \cos 3\theta$, ρ increases from 0 to a, while θ increases from 0 to $\frac{1}{6}\pi$; ρ decreases from a to $-a$, while θ increases from $\frac{1}{6}\pi$ to $\frac{1}{2}\pi$; ρ increases from $-a$ to $+a$, while θ increases from $\frac{1}{2}\pi$ to $\frac{5}{6}\pi$; and ρ decreases from a to 0, while θ increases from $\frac{5}{6}\pi$ to π. Further revolution of the radius vector in either direction would evidently retrace the three loops already found. The curve is that represented in Fig. 47.

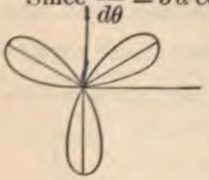

Fig. 47.

2. Trace the curve $\rho = a \sin 2\theta$.

The curve consists of four loops. From this and the previous example, we infer that the locus of $\rho = a \sin n\theta$ consists of n loops when n is odd, and $2n$ loops when n is even.

3. Trace $\rho = a \sin \frac{\theta}{2}$.

4. Trace the lituus $\rho = \frac{a}{\sqrt{\theta}}$.

5. Trace $\rho = a \cos\theta + b$, in which $a > b$.

MISCELLANEOUS EXAMPLES.

1. Examine $x^4 - axy^2 - ay^3 = 0$ for multiple points.

The origin, a triple point, is a cusp of the first species, through which a branch of the curve passes.

2. Examine $ax^3 + by^3 - c = 0$ for points of inflexion.

3. Prove that $(0, 0)$ is a multiple point of intersection on the curve $x^4 - a^2xy + b^2y^2 = 0$.

4. Examine $x^4 - a^2x^2 + a^3y = 0$ for points of inflexion.

Ans. $\left(\frac{a}{\sqrt{6}}, \frac{5a}{36}\right)$, and $\left(-\frac{a}{\sqrt{6}}, \frac{5a}{36}\right)$.

5. Examine $ay^2 - x^3 - bx^2 = 0$ for multiple points.

6. Examine $ay^2 - x^3 + bx^2 = 0$ for multiple points.

Ans. $(0, 0)$ is a conjugate point.

7. Trace the curve $y^2 = \frac{x^3 - a^3}{x + b}$.

8. Trace the folium of Descartes $y^3 - 3axy + x^3 = 0$.

9. Trace $(y - x^2)^2 = x^5$.

10. Trace $y^2(x - a) = x^3$.

11. Trace $\rho \cos\theta = a \cos 2\theta$.

CHAPTER XII.

CURVATURE, EVOLUTES, ENVELOPES, AND ORDER OF CONTACT.

152. If ϕ represent the inclination of any curve ab referred to rectangular axes, then ϕ will measure the *direction* of the curve with respect to the axis of x. At the point P, $\phi =$ angle XOR; and at P′, $\phi =$ XMP′; hence angle PRM $= \Delta\phi$, if arc PP′ $= \Delta s$, s representing the length of the curve.

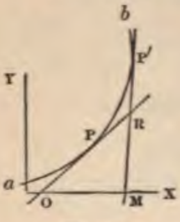

Fig. 48.

153. The **Curvature** of a curve at any point is the rate of change of its direction relative to that of its length.

Hence, *if* κ *represent the curvature of any curve,*

$$\kappa = \frac{d\phi}{ds}, \text{ or } \lim_{\Delta s \doteq 0}\left[\frac{\Delta\phi}{\Delta s}\right]. \qquad \S\, 31.$$

154. Curvature of a Circle. If ab (Fig. 48) be the arc of a circle whose radius is r, the angle PRM equals the angle subtended by the arc PP′ at its centre; and from § 40 we have

$$\text{angle PRM} = \frac{\text{arc PP}'}{r}, \text{ or } \frac{\Delta\phi}{\Delta s} = \frac{1}{r};$$

$$\therefore \frac{d\phi}{ds}, \text{ or } \kappa, = \frac{1}{r}. \qquad \S\, 17, \text{Cor. } 2.$$

Hence, *the curvature of any circle is equal to the reciprocal of its radius; and the curvatures of any two circles are inversely proportional to their radii.*

COR. *If* $r=1$, $\kappa=1$; that is, *the unit of curvature is the curvature of a circle whose radius is unity.*

155. *To find κ in terms of the differentials of* x *and* y.

$$\tan\phi = \frac{dy}{dx};$$

$$\therefore \sec^2\phi\, d\phi = \frac{d^2y}{dx};$$

$$\therefore d\phi = \frac{\dfrac{d^2y}{dx}}{1+\left(\dfrac{dy}{dx}\right)^2}, \text{ since } \sec^2\phi = 1+\left(\frac{dy}{dx}\right)^2.$$

$$\therefore \kappa\left[=\frac{d\phi}{ds}\right] = \frac{\dfrac{d^2y}{dx^2}}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}.$$

We take the positive value of the radical so that the curvature of a curve will be positive or negative according as $\dfrac{d^2y}{dx^2}$ is positive or negative; that is, according as the curve is concave upward or downward. The sign of curvature, however, is often neglected.

156. Radius of Curvature. As the radius of a circle varies from 0 to ∞, its curvature varies from ∞ to 0; hence there is always a circle whose curvature is equal to that of any curve at any point. A circle tangent to a curve, and having the same curvature as the curve at the point of contact, is called the **Circle of Curvature** of the curve at that point. Its radius and centre are the *radius of curvature* and the *centre of curvature* of the curve at that point. Hence, if R represent the radius of curvature of a curve, from §§ 154 and 155, we have

$$R = \frac{1}{\kappa} = \frac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}.$$

R will be positive or negative, according as the curve is concave upward or downward; but its sign is often neglected.

157. The *radius of curvature* in terms of *polar coördinates* can be found by transforming the value of R in § 156 to polar coördinates. We thus obtain

$$R = \frac{\left(\rho^2 + \frac{d\rho^2}{d\theta^2}\right)^{\frac{3}{2}}}{\rho^2 + 2\frac{d\rho^2}{d\theta^2} - \rho\frac{d^2\rho}{d\theta^2}} = \frac{N^3}{\rho^2 + 2\frac{d\rho^2}{d\theta^2} - \rho\frac{d^2\rho}{d\theta^2}},$$

in which N is the normal. See § 131, Ex. 3, and § 136, (7).

158. *The circle of curvature, in general, cuts the curve at the point of contact.*

For, on one side of the point of contact, the curve changes its direction more rapidly than the circle of curvature, and hence lies within the circle; while on the other side it changes its direction more slowly than the circle, and hence lies without the circle.

159. *At a point of maximum or minimum curvature,* the circle of curvature does not cut the curve; and conversely.*

For, on either side of a point of maximum curvature, the curve changes its direction more slowly than at this point; hence, on each side of this point, the curve lies without the circle of curvature at this point. On either side of a point of minimum curvature, the curve changes its direction more rapidly than at this point; hence, on each side of this point, the curve lies within the circle of curvature.

Since the conic sections are symmetrical with respect to normals at their vertices, it follows that their vertices must be points of maximum or minimum curvature.

EXAMPLES.

1. Find the curvature of the parabola $y^2 = 2px$.

Here $\frac{dy}{dx} = \frac{p}{y}$, and $\frac{d^2y}{dx^2} = -\frac{p^2}{y^3}$;

---

* By a maximum or minimum curvature, we mean a numerical maximum or minimum, the sign of curvature not being considered.

$$\therefore \kappa = \frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}} = -\frac{p^2}{y^3}\left(\frac{y^2}{y^2+p^2}\right)^{\frac{3}{2}} = \pm\frac{p^2}{(y^2+p^2)^{\frac{3}{2}}}.$$

The upper or lower sign is to be taken according as $-\frac{p^2}{y^3}$ is positive or negative.

At the vertex (0, 0), $\kappa = \frac{1}{p}$, which is evidently the maximum curvature of the parabola.

2. Neglecting its sign, find the curvature of the ellipse $a^2y^2 + b^2x^2 = a^2b^2$.

Here $\frac{dy}{dx} = -\frac{b^2x}{a^2y}$, and $\frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}$;

$$\therefore \kappa = \frac{b^4}{a^2y^3}\left(\frac{a^4y^2}{a^4y^2+b^4x^2}\right)^{\frac{3}{2}} = \frac{a^4b^4}{(a^4y^2+b^4x^2)^{\frac{3}{2}}}.$$

At the vertex $(a, 0)$, $\kappa = \frac{a}{b^2}$; at the vertex $(0, -b)$, $\kappa = \frac{b}{a^2}$; hence, the maximum curvature of the ellipse is $\frac{a}{b^2}$, and the minimum $\frac{b}{a^2}$.

3. Find the radius of curvature of the cycloid

$$x = r \text{ vers}^{1}\frac{y}{r} - \sqrt{2ry - y^2}.$$

Here $\frac{dy}{dx} = \frac{\sqrt{2ry-y^2}}{y}$, and $\frac{d^2y}{dx^2} = -\frac{r}{y^2}$;

$$\therefore R = \left(\frac{2r}{y}\right)^{\frac{3}{2}}\left(-\frac{y^2}{r}\right) = -2\sqrt{2ry},$$

which equals numerically twice the normal.

$$\kappa = \frac{1}{-2\sqrt{2ry}};$$

and $-\frac{1}{4r}$, the curvature at the highest point, is evidently the minimum curvature of the cycloid.

4. Find the curvature of $y = x^4 - 4x^3 - 18x^2$ at the origin. Find the abscissas of the points at which the curvature is 0.

Ans. $\kappa = -36$; $x = 3$ and -1.

5. Find the curvature of the logarithmic curve $y = a^x$.

Ans. $\kappa = \dfrac{my}{(m^2 + y^2)^{\frac{3}{2}}}$.

6. Find the numerical value of the radius of curvature of the cubical parabola $y^3 = a^2x$.

Ans. $R = \dfrac{(9y^4 + a^4)^{\frac{3}{2}}}{6a^4y}$.

7. Find the radius of curvature of the spiral of Archimedes $\rho = a\theta$.

Here $\dfrac{d\rho}{d\theta} = a$, and $\dfrac{d^2\rho}{d\theta^2} = 0$;

$$\therefore R = \frac{\left(\rho^2 + \frac{d\rho^2}{d\theta^2}\right)^{\frac{3}{2}}}{\rho^2 + 2\frac{d\rho^2}{d\theta^2} - \rho\frac{d^2\rho}{d\theta^2}} = \frac{(\rho^2 + a^2)^{\frac{3}{2}}}{\rho^2 + 2a^2} = a\frac{(1 + \theta^2)^{\frac{3}{2}}}{2 + \theta^2}.$$

8. Find the radius of curvature of the logarithmic spiral $\rho = a^\theta$.

Ans. $R = \rho\sqrt{1 + (\log a)^2}$.

Evolutes.

160. The **Evolute** of a given curve is the locus of the *centre of curvature* of the curve. The given curve is called the **Involute** of its evolute.

161. *To find the equation of the evolute of any given curve.*

Let C, (α, β), be the centre of curvature of the curve ab at any point P, (x, y). Then,

since $\text{BP} = \text{PC} \sin \text{BCP} = R \sin \text{KPH} = R\dfrac{dy}{ds}$,

and $\text{BC} = R \cos \text{BCP} = R\dfrac{dx}{ds}$;

$$\alpha = \text{OE} - \text{BP} = x - R\frac{dy}{ds}, \quad (1)$$

and $$\beta = \text{EP} + \text{BC} = y + R\frac{dx}{ds}. \quad (2)$$

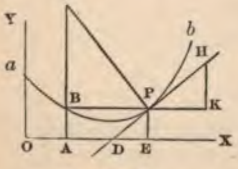

Fig. 49.

Substituting in (1) and (2) the values of R and ds, we have

$$\alpha = x - \frac{\left(1+\frac{dy^2}{dx^2}\right)\frac{dy}{dx}}{\frac{d^2y}{dx^2}}, \tag{3}$$

and
$$\beta = y + \frac{1+\frac{dy^2}{dx^2}}{\frac{d^2y}{dx^2}}. \tag{4}$$

By differentiating the equation of any given curve, and substituting the results in (3) and (4), α and β may be expressed in terms of x and y. If, between the equations thus obtained and that of the given curve, x and y be eliminated, the resulting equation between α and β will be the equation of the evolute.

EXAMPLES.

1. Find the equation of the evolute of the parabola.

Here $\frac{dy}{dx}=\frac{p}{y}$, and $\frac{d^2y}{dx^2}=-\frac{p^2}{y^3}$.

Substituting these values in (3) and (4) of § 161, and reducing, we have

$$\alpha = 3x+p, \quad \text{or} \quad x=\frac{\alpha-p}{3},$$

and
$$\beta = -\frac{y^3}{p^2}, \quad \text{or} \quad y=-\beta^{\frac{1}{3}}p^{\frac{2}{3}}.$$

Substituting these values of x and y in the equation of the parabola, we have

$$\beta^{\frac{2}{3}}p^{\frac{4}{3}} = \tfrac{2}{3}p(\alpha-p),$$

or
$$\beta^2 = \frac{8}{27p}(\alpha-p)^3,$$

which is the equation of the evolute of $y^2=2px$.

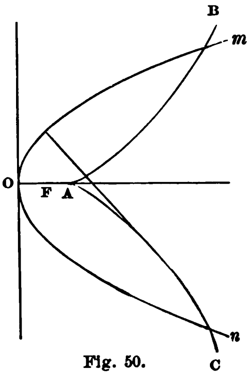

Fig. 50.

BAC (Fig. 50) is the evolute of the parabola *mon*.

2. Find the equation of the evolute of the ellipse.

Here $\dfrac{dy}{dx} = -\dfrac{b^2x}{a^2y}$, and $\dfrac{d^2y}{dx^2} = -\dfrac{b^4}{a^2y^3}$;

$$\therefore \alpha = \frac{(a^2-b^2)x^3}{a^4}, \text{ or } x = \left(\frac{a^4\alpha}{a^2-b^2}\right)^{\frac{1}{3}},$$

and $$\beta = -\frac{(a^2-b^2)y^3}{b^4}, \text{ or } y = -\left(\frac{b^4\beta}{a^2-b^2}\right)^{\frac{1}{3}}.$$

Substituting these values of x and y in $a^2y^2 + b^2x^2 = a^2b^2$, we obtain as the equation of the evolute of the ellipse,

$$(a\alpha)^{\frac{2}{3}} + (b\beta)^{\frac{2}{3}} = (a^2-b^2)^{\frac{2}{3}}.$$

3. Find the equation of the evolute of the cycloid

$$x = r\,\text{vers}^{-1}\frac{y}{r} - \sqrt{2ry - y^2}. \qquad (1)$$

Here $\dfrac{dy}{dx} = \dfrac{\sqrt{2ry - y^2}}{y}$, and $\dfrac{d^2y}{dx^2} = -\dfrac{r}{y^2}$;

$$\therefore y = -\beta, \text{ and } x = \alpha - 2\sqrt{-2r\beta - \beta^2}.$$

Substituting these values of x and y in (1), we obtain as the equation of the evolute of the cycloid,

$$\alpha = r\,\text{vers}^{1}\left(-\frac{\beta}{r}\right) + \sqrt{-2r\beta - \beta^2}. \qquad (2)$$

The locus of (2) is another cycloid equal to the given cycloid, the highest point being at the origin. For, if the cycloid OO_1M be referred to the axes O_1X_1 and O_1Y_1, and P be any point,

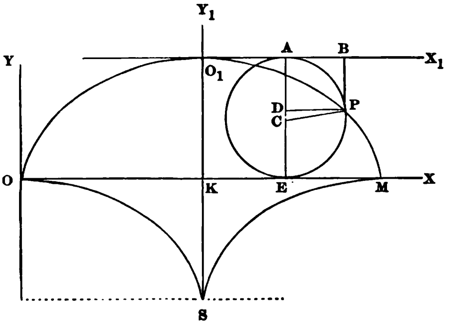

Fig. 51.

$$y = -BP,$$

and $$x = O_1B$$
$$= KE + DP. \qquad (3)$$

$$EM = \text{arc } EP;$$

$$\therefore \text{KE} = \text{arc AP} = r \operatorname{vers}^{-1}\frac{\text{AD}}{r} = r \operatorname{vers}^{-1}\left(\frac{-y}{r}\right). \tag{4}$$

$$\text{DP} = \sqrt{\text{AD}\cdot\text{DE}} = \sqrt{-y(2r+y)} = \sqrt{-2ry-y^2}. \tag{5}$$

From (3), (4), and (5), we obtain

$$x = r \operatorname{vers}^{-1}\left(\frac{-y}{r}\right) + \sqrt{-2ry-y^2}. \tag{6}$$

Comparing (2) with (6), we see that OS, the evolute of OO_1, must be equal to O_1M; that is, *the evolute of a cycloid is an equal cycloid.*

Properties of the Evolute.

162. *Any normal to the involute is a tangent to the evolute.*

The equation of the normal to $y=f(x)$ at (x', y') is

$$y - y' = -\frac{dx'}{dy'}(x-x'). \tag{1}$$

Let (α, β) be the centre of curvature of $y=f(x)$ at (x', y'); then (1) passes through (α, β), and we have

$$y' - \beta = -\frac{dx'}{dy'}(x'-\alpha); \tag{2}$$

$$\therefore x' - \alpha + \frac{dy'}{dx'}(y'-\beta) = 0. \tag{3}$$

If (x', y') move along the involute, (α, β) will move along the evolute, and α, β, and y' will be functions of x'. Differentiating (3) on this hypothesis, we have

$$dx' - d\alpha + \frac{dy'^2 - dy'd\beta}{dx'} + (y'-\beta)\frac{d^2y'}{dx'} = 0.$$

Dividing by dx', and rearranging terms, we have

$$1 + \frac{dy'^2}{dx'^2} + (y'-\beta)\frac{d^2y'}{dx'^2} - \frac{d\alpha}{dx'} - \frac{dy'd\beta}{dx'^2} = 0. \tag{4}$$

But, as (α, β) is on the evolute, we have by § 161, (4),

$$\beta = y' + \frac{1 + \frac{dy'^2}{dx'^2}}{\frac{d^2y'}{dx'^2}};$$

$$\therefore (y' - \beta)\frac{d^2y'}{dx'^2} + 1 + \frac{dy'^2}{dx'^2} = 0. \tag{5}$$

From (4) and (5), we obtain

$$-\frac{d\alpha}{dx'} - \frac{dy' d\beta}{dx'^2} = 0, \text{ or } -\frac{dx'}{dy'} = \frac{d\beta}{d\alpha}.$$

Hence (1) is tangent to the evolute at (α, β).

163. *Any continuous arc of the evolute is equal to the difference between the radii of curvature of the involute that are tangent to this arc at its extremities.*

If $(x - \alpha)^2 + (y - \beta)^2 = R^2$ be the circle of curvature of any curve at (x', y'), we have

$$(x' - \alpha)^2 + (y' - \beta)^2 = R^2. \tag{1}$$

Suppose (x', y') to move along the curve; then y', α, β, and R will be functions of x'.

Differentiating (1) on this hypothesis, we obtain

$$\begin{aligned}(x' - \alpha)dx' + (y' - \beta)dy' - (x' - \alpha)d\alpha \\ - (y' - \beta)d\beta = RdR.\end{aligned} \tag{2}$$

From equation (2) of § 162 we have

$$y' - \beta = -\frac{dx'}{dy'}(x' - \alpha); \tag{3}$$

$$\therefore y' - \beta = \frac{d\beta}{d\alpha}(x' - \alpha). \tag{4}$$

From (3),

$$(x' - \alpha)dx' + (y' - \beta)dy' = 0. \tag{5}$$

From (2) and (5),

$$(x' - \alpha)d\alpha + (y' - \beta)d\beta = -RdR. \tag{6}$$

From (1) and (4),

$$(x'-\alpha)^2\frac{d\alpha^2+d\beta^2}{d\alpha^2}=R^2. \tag{7}$$

From (4) and (6),

$$(x'-\alpha)\frac{d\alpha^2+d\beta^2}{d\alpha}=-R\,dR. \tag{8}$$

Squaring (8) and dividing by (7), we obtain

$$d\alpha^2+d\beta^2=dR^2.$$

But, s being the length of the evolute, we have

$$d\alpha^2+d\beta^2=ds^2;$$

$$\therefore ds=\pm dR;$$

that is, R increases or decreases as fast as s increases.

Hence, if the length of the evolute of the parabola (Fig. 52) be estimated from the point A, we have

arc AP = KP — OA.

Again, if the length of the evolute of the cycloid (Fig. 51) be estimated from O,

arc OS = SO$_1$ = $4r$;

hence the length of one branch of the cycloid is $8r$.

164. These two properties of the evolute enable us to regard any involute as traced by a point in a string unwound from its evolute. Thus, if on a pattern of CPAO one end of a string be fastened at C, and the string be then stretched along the right of CPA, the point of the string which reaches O, when carried around to the right, will trace the arc OKm as the string unwinds from the evolute.

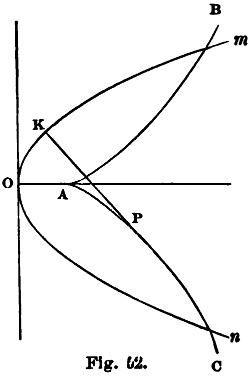

Fig. 52.

Since any point of the string beyond A will trace an involute of CA, it follows that, while a curve has but one evolute, it can have an infinite number of involutes.

Envelopes.

165. If, in the equation $f(x, y, a) = 0$, a series of different values be assigned to a, the equation will represent a series of curves differing in form, or in position, or in both these respects, but all belonging to the same class or family of curves.

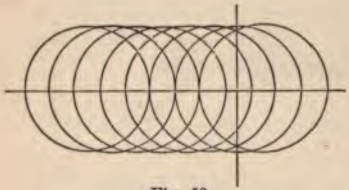

Fig. 53.

For example, if different values be assigned to a in $(x-a)^2 + y^2 = 25$, its loci will be a series of equal circles with their centres on the axis of x (Fig. 53).

The quantity a, which is constant for any one curve, but changes in passing from one curve to another, is called a **Variable Parameter.** Any two curves of a series that correspond to nearly equal values of the parameter usually intersect, and are called *consecutive* curves.

166. An **Envelope** is the locus of the *limiting positions* of the points of intersection of the consecutive curves of a series, as these curves approach indefinitely near each other.

167. *To find the equation of the envelope of a series of curves.*

Let $f(x, y, a) = u = 0$ (1)

and $f(x, y, a + \Delta a) = 0$ (2)

be the equations of any two consecutive curves of a series; then, at the points of intersection of (1) and (2), we evidently have

$$\frac{f(x, y, a + \Delta a) - f(x, y, a)}{\Delta a} = 0. \tag{3}$$

Passing to the limit as $\Delta\alpha \doteq 0$, we have, at the limiting positions of these intersections,

$$\frac{d}{d\alpha}f(x, y, \alpha) = 0, \text{ or } \frac{du}{d\alpha} = 0. \tag{4}$$

Since the coördinates of the points on the envelope satisfy both (4) and (1), its equation is found by eliminating α between these equations.

168. *The envelope is tangent to each curve of the series.*

Let $\phi(x, y)$ represent the value of α obtained from (4) of § 167; then, if $\alpha = \phi(x, y)$,

$$f(x, y, \alpha) = u = 0 \tag{1}$$

is the equation of the envelope.

Differentiating (1), α being variable, we obtain

$$\frac{du}{dx}dx + \frac{du}{dy}dy + \frac{du}{d\alpha}d\alpha = 0.$$

But, at any point on the envelope,

$$\frac{du}{d\alpha} = 0; \qquad \text{§ 167, (4)}$$

$$\therefore \frac{du}{dx}dx + \frac{du}{dy}dy = 0, \tag{2}$$

which gives the slope of the envelope at any point.

Differentiating $f(x, y, \alpha) = u = 0$, considering α constant, we have

$$\frac{du}{dx}dx + \frac{du}{dy}dy = 0, \tag{3}$$

which gives the slope of any individual curve at any point. From (2) and (3) we see that the envelope and any curve of the series have the same slope at their common point.

EXAMPLES.

1. Find the envelope of $(x - \alpha)^2 + y^2 = r^2$, in which α is a variable parameter.

Here $$f(x, y, \alpha)[=u]=(x-\alpha)^2+y^2-r^2=0; \qquad (1)$$

$$\therefore \frac{du}{d\alpha}=-2(x-\alpha)=0. \qquad (2)$$

From (1) and (2), $y=\pm r$; that is, the envelope is two lines parallel to the axis of x, as would be inferred from Fig. 53.

2. Find the envelope of $y=\alpha x+\frac{m}{\alpha}$, α being the variable parameter.

Here $$f(x, y, \alpha)[=u]=y-\alpha x-\frac{m}{\alpha}=0; \qquad (1)$$

$$\therefore \frac{du}{d\alpha}=-x+\frac{m}{\alpha^2}=0. \qquad (2)$$

Eliminating α between (1) and (2), we obtain $y^2=4mx$; that is, the envelope is a parabola whose latus-rectum is $4m$.

3. Find the envelope of the hypotenuse of a right-angled triangle of constant area.

Let α and β be the sides of the right triangle, and assume them as coördinate axes; then

$$\frac{x}{\alpha}+\frac{y}{\beta}=1$$

is the equation of the hypotenuse.

Let $c=$ the constant area; then

$$\alpha\beta=2c, \text{ or } \beta=\frac{2c}{\alpha}.$$

Hence $$f(x, y, \alpha)=\frac{x}{\alpha}+\frac{\alpha y}{2c}-1=0, \qquad (1)$$

and $$\frac{du}{d\alpha}=-\frac{x}{\alpha^2}+\frac{y}{2c}=0. \qquad (2)$$

Eliminating α between (1) and (2), we obtain $xy=\frac{1}{2}c$; that is, the envelope is an hyperbola to which the sides of the triangle are asymptotes.

4. Find the envelope of a system of concentric ellipses, the area and the directions of the axes being constant.

Let the equation of the ellipses be

$$\alpha^2 y^2 + \beta^2 x^2 = \alpha^2 \beta^2, \tag{1}$$

and c represent the constant area;

then $$c = \pi \alpha \beta. \tag{2}$$

Eliminating β between (1) and (2), we have

$$f(x, y, \alpha) = \alpha^2 y^2 + \frac{c^2 x^2}{\pi^2 \alpha^2} - \frac{c^2}{\pi^2} = 0; \tag{3}$$

$$\therefore \frac{du}{d\alpha} = 2\alpha y^2 - \frac{2c^2 x^2}{\pi^2 \alpha^3} = 0. \tag{4}$$

From (3) and (4), we obtain

$$xy = \pm \frac{c}{2\pi},$$

which are the equations of conjugate equilateral hyperbolas referred to their asymptotes.

169. Contact of Different Orders. Let $y = f(x)$ and $y = \phi(x)$ be any two curves referred to the same axes. If $f(a) = \phi(a)$, the curves have the point $[a, f(a)]$ in common. If $f(a) = \phi(a)$ and $f'(a) = \phi'(a)$, the curves are *tangent* at $[a, f(a)]$, and are said to have a *contact of the first order*. If $f(a) = \phi(a)$, $f'(a) = \phi'(a)$, and $f''(a) = \phi''(a)$, the two curves have the same curvature at their common point, and their *contact is of the second order*. If in addition, $f'''(a) = \phi'''(a)$, their contact is of the third order; and so on. *Thus, contact of the* n*th order imposes* n + 1 *conditions.*

170. *Two curves intersect or do not intersect at their point of contact, according as their order of contact is even or odd.*

Let $y = f(x)$ and $y = \phi(x)$ be any two curves having the point $[a, f(a)]$ in common. Let h be a very small increment of x By Taylor's formula, we have

$$f(a+h)=f(a)+f'(a)h+f''(a)\frac{h^2}{\underline{|2}}+f'''(a)\frac{h^3}{\underline{|3}}+\cdots \quad (1)$$

$$\phi(a+h)=\phi(a)+\phi'(a)h+\phi''(a)\frac{h^2}{\underline{|2}}+\phi'''(a)\frac{h^3}{\underline{|3}}+\cdots (2)$$

Subtracting (2) from (1), we obtain

$$f(a+h)-\phi(a+h)=h[f'(a)-\phi'(a)]+\frac{h^2}{\underline{|2}}[f''(a)-\phi''(a)]$$
$$+\frac{h^3}{\underline{|3}}[f'''(a)-\phi'''(a)]+\frac{h^4}{\underline{|4}}[f^{\text{iv}}(a)-\phi^{\text{iv}}(a)]+\cdots, \quad (3)$$

which gives the difference between the corresponding ordinates of the curves on each side of their common ordinate. If the contact is of an odd order, the first term of the second member of (3) which does not vanish contains an even power of h; hence the sign of the second member is the same whether h be positive or negative. Therefore, $y=f(x)$ lies either above or below $y=\phi(x)$ on both sides of their common point; and the curves do not intersect. But, if their contact is of an even order, the first term of the second member of (3) that does not vanish contains an odd power of h. Hence, in this case, the second member changes sign with h; and $y=f(x)$ lies above $y=\phi(x)$ on one side of their common point, and below it on the other; and the curves intersect.

Cor. At a point of maximum or minimum curvature, the circle of curvature has contact of the third order with the curve; for it does not cut the curve at such a point.

The following are obvious conclusions from equation (3):

(*a*) Two curves on each side of their common point are nearer, the higher their order of contact.

(*b*) If two curves have a contact of the nth order, no curve having with either of them a contact of a lower order can lie between them near their common point.

171. Osculating Curves. The curve of a given species, that has the highest order of contact possible with a given curve at any point, is called the *osculating curve* of that species.

Let $y = f(x)$ be the most general form of the equation of a curve of a given species, and suppose that it contains $n+1$ arbitrary constants. Upon the $n+1$ constants, $n+1$, and only $n+1$, independent conditions can be imposed. But, in order that $y = f(x)$ may have contact of the nth order with a given curve at a given point, $n+1$ conditions must be fulfilled by its constants; that is, these constants must have such values that $y = f(x)$ shall pass through the point, and the first n derivatives of its ordinate be equal to those of the given curve at this point (§ 169).

Hence, *as* $y = ax + b$ *has two constants, the osculating straight line has contact of the first order, and is a tangent.*

As $(x - a)^2 + (y - b)^2 = r^2$ *has three constants, the osculating circle has, in general, contact of the second order, and is the circle of curvature.*

The osculating parabola has contact of the third order. The osculating ellipse or hyperbola has contact of the fourth order.

MISCELLANEOUS EXAMPLES.

1. Find the curvature of the hyperbola.

$$\textit{Ans.}\quad \kappa = \pm \frac{a^4 b^4}{(b^4 x^2 + a^4 y^2)^{\frac{3}{2}}}.$$

2. Find the radius of curvature of the equilateral hyperbola $xy = \frac{1}{2} a^2$.

$$\textit{Ans.}\quad R = \pm \frac{(x^2 + y^2)^{\frac{3}{2}}}{a^2}.$$

3. Find the radius of curvature of $y^3 = 6x^2 + x^3$.

$$\textit{Ans.}\quad R = \frac{[y^4 + (4x + x^2)^2]^{\frac{3}{2}}}{8x^2 y} \text{ numerically.}$$

4. Find the radius of curvature of the lemniscate of Bernouilli, $\rho^2 = a^2 \cos 2\theta$.

$$\textit{Ans.}\quad R = \frac{a^2}{3\rho}.$$

5. Find the curvature of the cissoid $y^2 = \dfrac{x^3}{2a - x}$.

$$\textit{Ans.}\quad \kappa = \pm \frac{3(2a - x)^2}{a x^{\frac{1}{2}} (8a - 3x)^{\frac{3}{2}}}.$$

6. Find the equation of the evolute of the hyperbola.

Ans. $(a\alpha)^{\frac{2}{3}} - (b\beta)^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}}$.

7. Find the length of the evolute of the parabola in terms of the abscissas of its extremities.

$$R = \frac{(y^2 + p^2)^{\frac{3}{2}}}{p^2} = \frac{(2x + p)^{\frac{3}{2}}}{\sqrt{p}}. \qquad \S\ 159,\ \text{Ex. } 1.$$

$$\text{Arc AP (Fig. 52)} = \text{KP} - \text{OA} = \frac{(2x + p)^{\frac{3}{2}}}{\sqrt{p}} - p$$

$$= \frac{1}{\sqrt{p}}\left(\frac{2\alpha + p}{3}\right)^{\frac{3}{2}} - p. \qquad \S\ 161,\ \text{Ex. } 1.$$

8. Find the envelope of $y^2 = \alpha(x - \alpha)$, in which α is a variable parameter.

Ans. $y = \pm \frac{1}{2}x$.

9. One angle of a triangle is constant and fixed in position; find the envelope of the opposite side when the area is constant.

Ans. $xy = \dfrac{c}{2 \sin \omega}$, c being the constant area and ω the constant angle.

10. Find the envelope of the circles whose diameters are the double ordinates of the parabola $y^2 = 2px$.

Ans. $y^2 = p(p + 2x)$.

11. Find the envelope of a line of constant length a, whose extremities move along two fixed rectangular axes.

Ans. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

12. Find the envelope of the normals to the parabola $y^2 = 2px$.

Ans. $y^2 = \dfrac{8}{27p}(x - p)^3$, which is the evolute of the parabola, as it clearly should be.

13. Find the radius of curvature of the catenary $y = \dfrac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$.

Ans. $R = \dfrac{y^2}{a}$.

14. Find the radius of curvature of the cardioid $\rho=a(1-\cos\theta)$.

Ans. $R=\frac{(8a\rho)^{\frac{1}{2}}}{3}$.

15. Find the envelope of the series of ellipses $\frac{x^2}{a^2}+\frac{y^2}{(k-a)^2}=1$, a being a variable parameter.

Ans. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=k^{\frac{2}{3}}$.

16. Prove that the whole length of the evolute of the ellipse is $4\frac{a^3-b^3}{ab}$.

17. Find the radius of curvature of the tractrix, having given $\frac{dy}{dx}=-\frac{y}{\sqrt{a^2-y^2}}$. § 195, Ex. 5. *Ans.* $R=\frac{a}{y}(a^2-y^2)^{\frac{1}{2}}$.

18. Show that the evolute of a circle is its centre.

19. Prove that the catenary $\beta=\frac{a}{2}\left(\epsilon^{\frac{\alpha}{a}}-\epsilon^{-\frac{\alpha}{a}}\right)$ is the evolute of the tractrix $x=a\log\frac{a+(a^2-y^2)^{\frac{1}{2}}}{y}-(a^2-y^2)^{\frac{1}{2}}$. § 201, Ex. 1.

20. Find the equation of the envelope of $y=\alpha x+(a^2\alpha^2+b^2)^{\frac{1}{2}}$, in which α is a variable parameter.

Here the equation of the tangent is given, and that of the curve is required. *Ans.* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

21. Prove that $\frac{x^2}{a^2+b^2}+\frac{y^2}{b^2}=1$ is the envelope of the circles described on the double ordinates of the ellipse $a^2y^2+b^2x^2=a^2b^2$ as diameters.

CHAPTER XIII.

INTEGRATION OF RATIONAL FRACTIONS.

172. Decomposition of Rational Fractions. Any rational fraction whose numerator is not of a lower degree than its denominator can be separated by division into two parts, the one rational and entire, and the other a rational fraction whose numerator is of a lower degree than its denominator. For example,

$$\frac{x^4}{x^3+2x^2-x-2}=x-2+\frac{5x^2-4}{x^3+2x^2-x-2}.$$

Hence any rational differential can be considered as composed of an entire part and a fraction whose numerator is of a lower degree than its denominator. The entire part can be integrated by previous methods; and it is our present object to show that the fractional part, if not directly integrable, can be resolved into partial fractions which are integrable. These partial fractions differ in form, according as the simple factors of the denominator of the given fraction are:

I. Real and unequal.
II. Real and some of them equal.
III. Imaginary and unequal.
IV. Imaginary and some of them equal.

To show, in the simplest manner, how the decomposition and integration is to be effected, we shall apply the process to particular examples in each of the four cases.

173. Case I. *When the simple factors of the denominator are real and unequal, to every factor, as* $\mathrm{x}-\mathrm{a}$, *there corresponds a partial fraction of the form* $\dfrac{\mathrm{A}}{\mathrm{x}-\mathrm{a}}$.

Let it be required to find $\int \frac{(2x+3)\,dx}{x^3+x^2-2x}$.

The roots of $x^3+x^2-2x=0$ are 0, 1, and -2; hence, the factors of x^3+x^2-2x are x, $x-1$, and $x+2$.

Assume $$\frac{2x+3}{x(x-1)(x+2)}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+2}. \qquad (1)$$

Clearing (1) of fractions, we have

$$\begin{aligned} 2x+3 &= A(x-1)(x+2)+B(x+2)x \\ &\qquad + C(x-1)x \qquad (2) \\ &= (A+B+C)x^2+(A+2B-C)x-2A. \qquad (3) \end{aligned}$$

Equating the coefficients of like powers of x in (3), we have

$$A+B+C=0,\ A+2B-C=2, \text{ and } -2A=3. \qquad (4)$$

Solving equations (4), we find

$$A=-\tfrac{3}{2},\ B=\tfrac{5}{3}, \text{ and } C=-\tfrac{1}{6}. \qquad (5)$$

Substituting these values in (1), we obtain

$$\frac{2x+3}{x^3+x^2-2x}=-\frac{3}{2x}+\frac{5}{3(x-1)}-\frac{1}{6(x+2)}; \qquad (6)$$

$$\begin{aligned} \therefore \int \frac{(2x+3)\,dx}{x^3+x^2-2x} &= -\frac{3}{2}\int\frac{dx}{x}+\frac{5}{3}\int\frac{dx}{x-1}-\frac{1}{6}\int\frac{dx}{x+2} \\ &= -\tfrac{3}{2}\log x+\tfrac{5}{3}\log(x-1)-\tfrac{1}{6}\log(x+2)+\log c \\ &= \log\frac{c(x-1)^{\frac{5}{3}}}{x^{\frac{3}{2}}(x+2)^{\frac{1}{6}}}. \end{aligned}$$

Equation (6) is true; for the values of A, B, and C given in (5) satisfy (4), and hence make (3),* and therefore (1), an identical equation.

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* The principle used here is: *If* $A=A'$, $B=B'$, $C=C'$, *etc.*, $A+Bx+Cx^2+\cdots=A'+B'x+C'x^2+\cdots$ *is an identical equation.*

That the fraction has been correctly decomposed is proved by reasoning backward through the process. Equation (1) is not assumed as a basis of proof, but as a basis of operation.

The values of A, B, and C may be obtained from (2), as follows:

Making $x = 0$, we have $3 = -2A$; $\therefore A = -\frac{3}{2}$.
Making $x = 1$, we have $5 = 3B$; $\therefore B = \frac{5}{3}$.
Making $x = -2$, we have $-1 = 6C$; $\therefore C = -\frac{1}{6}$.

EXAMPLES.

1. Find $\int \frac{(x-1)dx}{x^2+6x+8}$. *Ans.* $\log \frac{(x+4)^{\frac{5}{2}}c}{(x+2)^{\frac{3}{2}}}$.

2. $\int \frac{(5x+1)dx}{x^2+x-2}$. $\log[(x-1)^2(x+2)^3c]$.

3. $\int \frac{(x^2+x-1)dx}{x^3+x^2-6x}$. $\log[x^{\frac{1}{6}}(x-2)^{\frac{1}{2}}(x+3)^{\frac{1}{3}}c]$.

4. $\int \frac{(x^2-1)dx}{x^2-4}$. $x + \log\left(\frac{x-2}{x+2}\right)^{\frac{3}{4}} + C$.

174. CASE II. *When some of the simple factors of the denominator are real and equal, to every set of equal factors, as* $(x-a)^n$, *there corresponds a series of* n *partial fractions of the form* $\frac{A}{(x-a)^n} + \frac{B}{(x-a)^{n-1}} + \cdots + \frac{L}{x-a}$.

Let it be required to find $\int \frac{dx}{(x-1)^2(x+1)}$.

Assume
$$\frac{1}{(x-1)^2(x+1)} = \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{x+1}. \qquad (1)$$

Clearing (1) of fractions, we have
$$\begin{aligned} 1 &= A(x+1) + B(x^2-1) + C(x-1)^2 \\ &= (B+C)x^2 + (A-2C)x + A - B + C. \qquad (2) \end{aligned}$$

Equating the coefficients of like powers of x, we obtain
$$B + C = 0,\ A - 2C = 0, \text{ and } A - B + C = 1. \qquad (3)$$

Whence $A = \frac{1}{2}$, $C = \frac{1}{4}$, and $B = -\frac{1}{4}$. (4)

Substituting these values in (1), and integrating, we have

$$\int\frac{dx}{(x-1)^2(x+1)}$$

$$=\int\frac{dx}{2(x-1)^2}-\int\frac{dx}{4(x-1)}+\int\frac{dx}{4(x+1)} \qquad (5)$$

$$=\log\left(\frac{x+1}{x-1}\right)^{\frac{1}{4}}-\frac{1}{2(x-1)}+C.$$

Equation (5) is true; for the values of A, B, and C given in (4) satisfy (3), and hence make (2), and therefore (1), an identical equation.

EXAMPLES.

1. Find $\int\frac{(2x-5)\,dx}{(x+3)(x+1)^2}$. $\quad \frac{7}{2(x+1)}+\frac{11}{4}\log\frac{x+1}{x+3}+C.$

2. $\int\frac{x^2dx}{x^3+5x^2+8x+4}$. $\quad \frac{4}{x+2}+\log(x+1)+C.$

3. $\int\frac{(2x^3+7x^2+6x+2)\,dx}{x^4+3x^3+2x^2}$. $\quad \log\left[x(x+1)\left(\frac{x}{x+2}\right)^{\frac{1}{2}}\right]-\frac{1}{x}+C.$

175. When the simple factors of its denominator are imaginary, a fraction may then also be decomposed into partial fractions of the forms in Cases I. and II.; but, since the integrals obtained from these would involve the logarithms of imaginaries, we seek other forms for the partial fractions.

Since the imaginary roots of an equation always occur in conjugate pairs, the imaginary factors of the denominator will occur in pairs whose products are real quadratic factors of the form $(x-a)^2+b^2$. Hence, when its simple factors are imaginary, the denominator can be resolved into real quadratic factors.

CASE III. *When some of the simple factors of the denominator are imaginary and unequal, to every factor of the form* $(\mathrm{x}-\mathrm{a})^2+\mathrm{b}^2$ *there corresponds a partial fraction of the form* $\frac{\mathrm{Ax}+\mathrm{B}}{(\mathrm{x}-\mathrm{a})^2+\mathrm{b}^2}$.

Let it be required to find $\int \frac{x^2 dx}{x^4 + x^2 - 2}$.

Assume
$$\frac{x^2}{(x+1)(x-1)(x^2+2)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+2}. \qquad (1)$$

Clearing (1) of fractions, we obtain

$$\begin{aligned} x^2 &= A(x-1)(x^2+2) + B(x+1)(x^2+2) \\ &\quad + (Cx+D)(x^2-1) \\ &= (A+B+C)x^3 + (B-A+D)x^2 \\ &\quad + (2A+2B-C)x + 2B - 2A - D. \qquad (2) \end{aligned}$$

Equating coefficients of like powers of x, we have

$$\left.\begin{array}{ll} A+B+C=0, & B-A+D=1, \\ 2A+2B-C=0, & 2B-2A-D=0. \end{array}\right\} \qquad (3)$$

Whence $A = -\frac{1}{6}$, $B = \frac{1}{6}$, $C = 0$, $D = \frac{2}{3}$. $\qquad (4)$

Hence
$$\begin{aligned} \int \frac{x^2 dx}{x^4+x^2-2} &= -\frac{1}{6}\int \frac{dx}{x+1} + \frac{1}{6}\int \frac{dx}{x-1} + \frac{2}{3}\int \frac{dx}{x^2+2} \qquad (5) \\ &= \frac{1}{6}\log\frac{x-1}{x+1} + \frac{1}{3}\sqrt{2}\tan^{-1}\frac{x}{\sqrt{2}} + C. \end{aligned}$$

Equation (5) is true; for the values of A, B, C, and D given in (4) satisfy (3), and hence make (2), and therefore (1), an identical equation.

EXAMPLES.

1. Find $\int \frac{x\,dx}{(x+1)(x^2+1)}$.

Ans. $\frac{1}{2}\tan^{-1}x + \frac{1}{2}\log\frac{(x^2+1)^{\frac{1}{2}}}{x+1} + C.$

2. $\int \frac{x^3 dx}{x^4 + 3x^2 + 2}$. $\qquad \log\frac{(x^2+2)c}{(x^2+1)^{\frac{1}{2}}}.$

3. $\int \frac{x^2\,dx}{(x-1)^2(x^2+1)}$. $\qquad -\frac{1}{2(x-1)} + \frac{1}{4}\log\frac{x^2-2x+1}{x^2+1} + C.$

4. $\displaystyle\int\frac{x^2dx}{1-x^4}.$ $\displaystyle\frac{1}{4}\log\frac{1+x}{1-x}-\frac{1}{2}\tan^{-1}x+C.$

5. $\displaystyle\int\frac{x^2dx}{x^3+x^2+x+1}.$

$$\tfrac{1}{2}\log(x+1)+\tfrac{1}{4}\log(x^2+1)-\tfrac{1}{2}\tan^{-1}x+C.$$

6. $\displaystyle\int\frac{dx}{x^3+1}.$

$$\int\frac{dx}{x^3+1}=\frac{1}{3}\int\frac{dx}{x+1}-\frac{1}{3}\int\frac{(x-2)dx}{x^2-x+1};$$

$$\int\frac{(x-2)dx}{x^2-x+1}=\frac{1}{2}\int\frac{(2x-1)dx}{x^2-x+1}-\frac{1}{2}\int\frac{3\,dx}{x^2-x+1};$$

$$\therefore\int\frac{dx}{x^3+1}=\frac{1}{6}\log\frac{(1+x)^2}{x^2-x+1}+\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}+C.$$

7. $\displaystyle\int\frac{dx}{1-x^3}.$ $\displaystyle\frac{1}{6}\log\frac{x^2+x+1}{x^2-2x+1}+\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}+C.$

176. Case IV. *When some of the simple factors of the denominator are imaginary and equal, to every set of equal quadratic factors of the form* $[(x-a)^2+b^2]^n$ *there corresponds a series of* n *partial fractions of the form*

$$\frac{Ax+B}{[(x-a)^2+b^2]^n}+\frac{Cx+D}{[(x-a)^2+b^2]^{n-1}}+\cdots+\frac{Lx+M}{(x-a)^2+b^2}.$$

In any example under this case, by clearing the assumed equation of fractions, and equating the coefficients of like powers of x, we should, as in the first three cases, evidently obtain as many simple equations as there are indeterminate quantities; and the values of A, B, C, etc., determined by these equations would make the assumed equation an identical one.

It remains to be proved that we can always integrate a fraction of the form $\dfrac{(Ax+B)dx}{[(x-a)^2+b^2]^n}$.

Let $z = x - a$; then $x = z + a$, $dx = dz$, and

$$\int \frac{(Ax+B)\,dx}{[(x-a)^2+b^2]^n} = \int \frac{(Az+Aa+B)\,dz}{(z^2+b^2)^n}$$

$$= \int \frac{Az\,dz}{(z^2+b^2)^n} + \int \frac{(Aa+B)\,dz}{(z^2+b^2)^n}$$

$$= \frac{-A}{2(n-1)(z^2+b^2)^{n-1}} + \int \frac{(Aa+B)\,dz}{(z^2+b^2)^n}.$$

The method of finding the integral of the last term is given in § 185.

EXAMPLES.

1. Find $\int \frac{(x^3+x-1)\,dx}{(x^2+2)^2}$.

Ans. $\frac{1}{2}\log(x^2+2) + \frac{1}{2(x^2+2)} - \int \frac{dx}{(x^2+2)^2}$.

Assume $\frac{x^3+x-1}{(x^2+2)^2} = \frac{Ax+B}{(x^2+2)^2} + \frac{Cx+D}{x^2+2}$.

2. $\int \frac{(x^2-x+1)\,dx}{x^3+x^2+x+1}$. $\qquad \log \frac{(x+1)^{\frac{3}{2}}}{(x^2+1)^{\frac{1}{4}}} - \frac{1}{2}\tan^{-1}x + C.$

3. $\int \frac{dx}{x^3-x^2+2x-2}$. $\qquad \log \frac{(x-1)^{\frac{1}{3}}}{(x^2+2)^{\frac{1}{6}}} - \frac{1}{3\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C.$

4. $\int \frac{(2-3x^2)\,dx}{(x+2)^3}$. $\qquad \frac{5}{(x+2)^2} - \frac{12}{x+2} - 3\log(x+2) + C.$

CHAPTER XIV.

INTEGRATION BY RATIONALIZATION.

177. It has been shown that any rational differential is integrable; hence an irrational differential which does not belong to a known form can be integrated, if we can *rationalize* it; that is, if we can find its *equivalent rational differential* in terms of a new variable which is some definite function of the given variable.

178. *A differential containing no surd but those of the form* $x^{\frac{c}{d}}$ *can be rationalized by assuming* $x = z^n$, *in which* n *is the least common multiple of all the denominators of the several fractional exponents of* x.

For, if $x = z^n$, the values of x, dx, and each of the surds, will be rational in terms of z; hence the function of z obtained by substituting these values in the given function will be rational.

EXAMPLES.

1. Find $\int \frac{x^{\frac{1}{2}} - x^{\frac{2}{3}}}{2x^{\frac{1}{6}}}\,dx.$ *Ans.* $\frac{3}{8}x^{\frac{4}{3}} - \frac{1}{3}x^{\frac{3}{2}} + C.$

Assume $x = z^6$; then

$$x^{\frac{1}{2}} = z^3,\quad x^{\frac{2}{3}} = z^4,\quad x^{\frac{1}{6}} = z, \text{ and } dx = 6z^5dz;$$

$$\therefore \int \frac{x^{\frac{1}{2}} - x^{\frac{2}{3}}}{2x^{\frac{1}{6}}}dx = \int \frac{z^3 - z^4}{2z} 6z^5dz = 3\int (z^7 - z^8)dz$$

$$= \tfrac{3}{8}z^8 - \tfrac{1}{3}z^9 + C = \tfrac{3}{8}x^{\frac{4}{3}} - \tfrac{1}{3}x^{\frac{3}{2}} + C.$$

2. $\int \frac{x^{\frac{1}{3}}dx}{x^{\frac{1}{2}} + 2x^{\frac{2}{3}}}.$ $\frac{3}{4}x^{\frac{2}{3}} - \frac{1}{2}x^{\frac{1}{2}} + \frac{3}{8}x^{\frac{1}{3}} - \frac{3}{8}x^{\frac{1}{6}} + \frac{3}{16}\log(1+2x^{\frac{1}{6}}) + C.$

179. *A differential containing no surd except* a + bx *affected with fractional exponents can be rationalized by assuming* a + bx = z^n, *in which* n *is the least common multiple of the denominators of the several fractional exponents.*

For, if $a + bx = z^n$, the values of x, dx, and each of the surds, will be rational in terms of z.

EXAMPLES.

1. Find $\int \frac{dx}{(1+x)^{\frac{3}{2}} + (1+x)^{\frac{1}{2}}}$. *Ans.* $2\tan^{-1}(1+x)^{\frac{1}{2}} + C$.

Assume $1 + x = z^2$; then

$$(1+x)^{\frac{3}{2}} = z^3,$$

$$(1+x)^{\frac{1}{2}} = z, \text{ and } dx = 2z\,dz;$$

$$\therefore \int \frac{dx}{(1+x)^{\frac{3}{2}} + (1+x)^{\frac{1}{2}}} = \int \frac{2z\,dz}{z^3 + z} = 2\int \frac{dz}{z^2+1}$$

$$= 2\tan^{-1}z + C = 2\tan^{-1}(1+x)^{\frac{1}{2}} + C.$$

2. $\int \frac{x\,dx}{(a+bx)^{\frac{3}{2}}}$. $\frac{2(2a+bx)}{b^2\sqrt{a+bx}} + C$.

3. $\int \frac{y\,dy}{(2r-y)^{\frac{1}{2}}}$. $-\frac{2}{3}(4r+y)(2r-y)^{\frac{1}{2}} + C$.

4. $\int x(a+x)^{\frac{1}{3}}dx$. $\frac{3}{28}(4x-3a)(a+x)^{\frac{4}{3}} + C$.

5. $\int \frac{dx}{x\sqrt{a+bx}}$. $\frac{1}{\sqrt{a}}\log\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}} + \log c$.

6. $\int \frac{x^2\,dx}{(1+x)^{\frac{2}{3}}}$. $3(1+x)^{\frac{1}{3}}\left[\frac{(1+x)^2}{7} + \frac{1-x}{2}\right] + C$.

180. *A differential containing no surd except* $\sqrt{a + bx + x^2}$ *can be rationalized by assuming* $\sqrt{a + bx + x^2} = z - x$.

Let $\sqrt{a+bx+x^2}=z-x;$

then $a+bx=z^2-2zx,$

$$x=\frac{z^2-a}{b+2z},$$

$$dx=\frac{2(z^2+bz+a)dz}{(b+2z)^2},$$

and $\sqrt{a+bx+x^2}\,[=z-x]=\frac{z^2+bz+a}{b+2z}.$

Hence, as the values of x, dx, and $\sqrt{a+bx+x^2}$ expressed in terms of z are rational, the given differential when expressed in terms of z must be rational.

181. *A differential containing no surd except* $\sqrt{\mathrm{a}+\mathrm{bx}-\mathrm{x}^2}$ *can be rationalized by assuming*

$$\sqrt{\mathrm{a}+\mathrm{bx}-\mathrm{x}^2}\,[=\sqrt{(\mathrm{x}-\beta)(\gamma-\mathrm{x})}]=(\mathrm{x}-\beta)\mathrm{z},$$

in which β *and* γ *are the roots of* $\mathrm{x}^2-\mathrm{bx}-\mathrm{a}=0.$

Let $\sqrt{a+bx-x^2}\,[=\sqrt{(x-\beta)(\gamma-x)}]=(x-\beta)z;$

then $\gamma-x=(x-\beta)z^2,$

$$x=\frac{\beta z^2+\gamma}{z^2+1},$$

$$dx=\frac{2(\beta-\gamma)z\,dz}{(z^2+1)^2},$$

and $\sqrt{a+bx-x^2}\,[=(x-\beta)z]=\frac{(\gamma-\beta)z}{z^2+1}.$

Hence, as the values of x, dx, and $\sqrt{a+bx-x^2}$ expressed in terms of z are rational, the differential when expressed in terms of z must be rational.

EXAMPLES.

1. Find $\int\frac{dx}{\sqrt{1+x+x^2}}$. *Ans.* $\log[(2x+1+2\sqrt{1+x+x^2})c].$

Assume $\sqrt{1+x+x^2}=z-x$;

then $$x=\frac{z^2-1}{2z+1},$$

$$dx=\frac{2(z^2+z+1)dz}{(2z+1)^2},$$

and $$\sqrt{1+x+x^2}[=z-x]=\frac{z^2+z+1}{2z+1}.$$

$$\therefore \int\frac{dx}{\sqrt{1+x+x^2}}=\int\frac{2\,dz}{2z+1}=\log[(2z+1)c]$$

$$=\log[(2x+1+2\sqrt{1+x+x^2})c].$$

2. $\displaystyle\int\frac{dx}{\sqrt{2-x-x^2}}.$

The roots of $x^2+x-2=0$ are -2 and 1;

$$\therefore 2-x-x^2=(x+2)(1-x).$$

Assume $\sqrt{2-x-x^2}[=\sqrt{(x+2)(1-x)}]=(x+2)z$;

then $$1-x=(x+2)z^2,$$

$$x=\frac{1-2z^2}{z^2+1},$$

$$dx=\frac{-6z\,dz}{(z^2+1)^2},$$

and $$\sqrt{2-x-x^2}=\frac{3z}{z^2+1}.$$

$$\therefore \int\frac{dx}{\sqrt{2-x-x^2}}=\int\frac{-2\,dz}{z^2+1}=2\cot^{-1}z+C$$

$$=2\cot^{-1}\left(\frac{1-x}{x+2}\right)^{\frac{1}{2}}+C.$$

3. $\displaystyle\int\frac{dx}{\sqrt{5x-6-x^2}}.$ $\qquad 2\cot^{-1}\left(\frac{3-x}{x-2}\right)^{\frac{1}{2}}+C.$

4. $\displaystyle\int\frac{\sqrt{2x+x^2}\,dx}{x^2}.$ $\qquad \log(x+1+\sqrt{2x+x^2})-\frac{4}{x+\sqrt{2x+x^2}}+C.$

182. Binomial Differentials. Differentials of the form

$$x^m(a + bx^n)^p dx,$$

in which m, n, and p represent any numbers, are called *binomial differentials*. When p is a whole number, the binomial factor can be expanded, and the differential exactly integrated by previous methods. In what follows, p is regarded as fractional; and, in the next section, we will represent it by $\frac{r}{s}$, r and s being whole numbers.

183. Conditions of Rationalization of $x^m(a + bx^n)^{\frac{r}{s}}dx$.

I. Assume $a + bx^n = z^s$;

then $$(a + bx^n)^{\frac{r}{s}} = z^r; \tag{1}$$

$$x = \left(\frac{z^s - a}{b}\right)^{\frac{1}{n}}, \quad x^m = \left(\frac{z^s - a}{b}\right)^{\frac{m}{n}}; \tag{2}$$

and $$dx = \frac{s}{bn} z^{s-1}\left(\frac{z^s - a}{b}\right)^{\frac{1}{n} - 1} dz. \tag{3}$$

Multiplying (1), (2), and (3) together, we obtain

$$x^m(a + bx^n)^{\frac{r}{s}} dx = \frac{s}{bn} z^{r+s-1}\left(\frac{z^s - a}{b}\right)^{\frac{m+1}{n} - 1} dz. \tag{4}$$

The second member of (4) is rational, and therefore integrable, when $\frac{m+1}{n}$ is a whole number or zero.

II. Assume $a + bx^n = z^s x^n$;

then $$x^n = a(z^s - b)^{-1}; \tag{1}$$

$$x = a^{\frac{1}{n}}(z^s - b)^{-\frac{1}{n}}, \quad x^m = a^{\frac{m}{n}}(z^s - b)^{-\frac{m}{n}}; \tag{2}$$

and $$dx = -\frac{s}{n} a^{\frac{1}{n}} z^{s-1}(z^s - b)^{-\frac{1}{n} - 1} dz. \tag{3}$$

Multiplying (1) by b, and adding a, we obtain

$$a + bx^n = \frac{az^s}{z^s - b};$$

$$\therefore (a+bx^n)^{\frac{r}{s}} = a^{\frac{r}{s}}(z^s-b)^{-\frac{r}{s}}z^r. \qquad (4)$$

Multiplying (2), (3), and (4) together, we have

$$x^m(a+bx^n)^{\frac{r}{s}}dx$$

$$= -\frac{s}{n}a^{\frac{m+1}{n}+\frac{r}{s}}(z^s-b)^{-\left(\frac{m+1}{n}+\frac{r}{s}+1\right)}z^{r+s-1}dz. \qquad (5)$$

The second member of (5) is rational and integrable, when $\frac{m+1}{n}+\frac{r}{s}$ is a whole number or zero.

Hence, $x^m(a+bx^n)^{\frac{r}{s}}dx$ can be integrated *by rationalization*.

I. When $\frac{m+1}{n}$[*] is a *whole number* or *zero*, by assuming $a+bx^n=z^s$.

II. When $\frac{m+1}{n}+\frac{r}{s}$ is a *whole number* or *zero*, by assuming $a+bx^n=z^sx^n$.

EXAMPLES.

1. Find $\int x^3(a+bx^2)^{-\frac{1}{2}}dx$. *Ans.* $\frac{bx^2-2a}{3b^2}(a+bx^2)^{\frac{1}{2}}+C$.

Here $\frac{m+1}{n}$ is a whole number, and $s=2$; hence we assume

$$a+bx^2=z^2;$$

$$\therefore (a+bx^2)^{-\frac{1}{2}}=z^{-1}; \qquad (1)$$

$$x=\left(\frac{z^2-a}{b}\right)^{\frac{1}{2}}, \quad x^3=\left(\frac{z^2-a}{b}\right)^{\frac{3}{2}}; \qquad (2)$$

---

[*] When $\frac{m+1}{n}-1$ is a negative integer, or $\frac{m+1}{n}+\frac{r}{s}+1$ is a positive integer, the exponent of z^s-b being negative, the given differential will be reduced to a rational fraction whose integral may be obtained by the method of Chapter XIII. But, as this method usually gives a complicated result, it is generally expedient in such cases to integrate by using the formulas of reduction given in § 185.

and $$dx=\frac{z}{b}\left(\frac{b}{z^2-a}\right)^{\frac{1}{2}}dz. \qquad (3)$$

Multiplying (1), (2), and (3) together, we obtain

$$\int x^3(a+bx^2)^{-\frac{1}{2}}dx=\frac{1}{b^2}\int(z^2-a)\,dz$$

$$=\frac{1}{3b^2}(z^3-3az)+C=\frac{bx^2-2a}{3b^2}(a+bx^2)^{\frac{1}{2}}+C.$$

2. $\displaystyle\int\frac{x^3\,dx}{(2-3x^2)^{\frac{3}{2}}}.$ $\qquad\displaystyle\frac{4-3x^2}{9(2-3x^2)^{\frac{1}{2}}}+C.$

3. $\displaystyle\int x^{-4}(1+x^2)^{-\frac{1}{2}}dx.$ $\qquad\displaystyle\frac{(2x^2-1)(1+x^2)^{\frac{1}{2}}}{3x^3}+C.$

Here $\dfrac{m+1}{n}+\dfrac{r}{s}$ is a whole number, and $s=2$; hence we assume

$$1+x^2=z^2x^2;\ \therefore x^2=(z^2-1)^{-1},$$

$$x=(z^2-1)^{-\frac{1}{2}},\ x^{-4}=(z^2-1)^2, \qquad (1)$$

$$dx=-(z^2-1)^{-\frac{3}{2}}z\,dz; \qquad (2)$$

and $$(1+x^2)^{-\frac{1}{2}}=\left(1+\frac{1}{z^2-1}\right)^{-\frac{1}{2}}=z^{-1}(z^2-1)^{-\frac{1}{2}}. \qquad (3)$$

Multiplying (1), (2), and (3) together, we have

$$\int\frac{dx}{x^4(1+x^2)^{\frac{1}{2}}}=-\int(z^2-1)\,dz$$

$$=z-\frac{z^3}{3}+C=\frac{(2x^2-1)(1+x^2)^{\frac{1}{2}}}{3x^3}+C.$$

4. $\displaystyle\int\frac{dx}{x(a^2+x^2)^{\frac{1}{2}}}.$ $\qquad\displaystyle\frac{1}{a}\log\frac{x}{\sqrt{a^2+x^2}+a}+\log c.$

5. $\displaystyle\int\frac{dx}{x(a^2-x^2)^{\frac{1}{2}}}.$ $\qquad\displaystyle\frac{1}{a}\log\frac{x}{\sqrt{a^2-x^2}+a}+\log c.$

6. $\displaystyle\int\frac{a\,dx}{(1+x^2)^{\frac{3}{2}}}.$ $\qquad\displaystyle\frac{ax}{(1+x^2)^{\frac{1}{2}}}+C.$

MISCELLANEOUS EXAMPLES.

1. Find $\int \frac{(1-x^{\frac{1}{2}})\,dx}{1-x^{\frac{1}{3}}}$.

Ans. $6[\frac{1}{7}x^{\frac{7}{6}}+\frac{1}{5}x^{\frac{5}{6}}-\frac{1}{4}x^{\frac{2}{3}}+\frac{1}{3}x^{\frac{1}{2}}-\frac{1}{2}x^{\frac{1}{3}}+x^{\frac{1}{6}}-\log(1+x^{\frac{1}{6}})]+C.$

2. $\int \frac{dx}{(1-x)(1+x)^{\frac{1}{2}}}$. $\qquad \frac{1}{\sqrt{2}}\log\frac{\sqrt{2}+\sqrt{1+x}}{\sqrt{2}-\sqrt{1+x}}+\log c.$

3. $\int \frac{dx}{(2+x)(1+x)^{\frac{1}{2}}}$. $\qquad 2\tan^{-1}(1+x)^{\frac{1}{2}}+C.$

4. $\int (a+bx)^{\frac{3}{2}}x\,dx.$ $\qquad \frac{2(a+bx)^{\frac{5}{2}}}{b^2}\left(\frac{a+bx}{7}-\frac{a}{5}\right)+C.$

5. $\int \frac{x\,dx}{(a+bx)^{\frac{1}{3}}}$. $\qquad \frac{3}{b^2}(a+bx)^{\frac{2}{3}}\left(\frac{a+bx}{5}-\frac{a}{2}\right)+C.$

6. $\int \frac{dx}{x(bx-a)^{\frac{1}{2}}}$. $\qquad \frac{2}{\sqrt{a}}\tan^{-1}\left(\frac{bx-a}{a}\right)^{\frac{1}{2}}+C.$

7. $\int \frac{dx}{(1+x)\sqrt{2+x-x^2}}$. $\qquad -\frac{2}{3}\left(\frac{2-x}{x+1}\right)^{\frac{1}{2}}+C.$

8. $\int \frac{dx}{(1+x)\sqrt{1+x+x^2}}$. $\qquad \log\left(\frac{x+\sqrt{1+x+x^2}}{2+x+\sqrt{1+x+x^2}}c\right).$

9. $\int x^3(a+bx^2)^{\frac{3}{2}}dx.$ $\qquad (a+bx^2)^{\frac{5}{2}}\left(\frac{5bx^2-2a}{35b^2}\right)+C.$

10. $\int \frac{dx}{(1+x^2)^{\frac{5}{2}}}$. $\qquad \frac{x(2x^2+3)}{3(1+x^2)^{\frac{3}{2}}}+C.$

11. $\int \frac{x^2\,dx}{(a+bx^2)^{\frac{5}{2}}}$. $\qquad \frac{x^3}{3a(a+bx^2)^{\frac{3}{2}}}+C.$

12. $\int \frac{dx}{x^2(a+bx^2)^{\frac{3}{2}}}$. $\qquad -\frac{a+2bx^2}{a^2x(a+bx^2)^{\frac{1}{2}}}+C.$

CHAPTER XV.

INTEGRATION BY PARTS AND BY SERIES.

184. If u and v be any functions of x, we have

$$d(uv) = udv + vdu.$$

Integrating and transposing, we have

$$\int udv = uv - \int vdu. \tag{1}$$

Equation (1) is the *formula for integration by parts.* It reduces the integration of udv to that of vdu; and, by its application, many useful *formulas of reduction* are obtained.

Examples.

1. Find $\int x \log x dx$. *Ans.* $\frac{1}{2}x^2 \log x - \frac{1}{4}x^2 + C.$

Assume $u = \log x$, and $dv = x dx$;

then $du = \frac{dx}{x}$, and $v = \frac{x^2}{2}$.

Substituting these values in formula (1), we have

$$\int x \log x dx = \frac{1}{2}x^2 \log x - \int \frac{1}{2} x dx$$
$$= \frac{1}{2}x^2 \log x - \frac{1}{4}x^2 + C.$$

In each example, the values of u and dv must be so assumed that vdu is a known form, or nearer one than udv.

2. $\int \log x dx$. $x(\log x - 1) + C.$

3. $\int xe^{ax} dx$. $e^{ax}\left(\frac{x}{a} - \frac{1}{a^2}\right) + C.$

Assume $dv = e^{ax} dx$.

4. $\int x^n \log x\,dx.$ $\qquad \dfrac{x^{n+1}}{n+1}\left(\log x - \dfrac{1}{n+1}\right) + C.$

5. $\int \sin^{-1} x\,dx.$ $\qquad x\sin^{-1}x + (1-x^2)^{\frac{1}{2}} + C.$

6. $\int \tan^{-1} x\,dx.$ $\qquad x\tan^{-1}x - \log(1+x^2)^{\frac{1}{2}} + C.$

7. $\int x^3(a-x^2)^{\frac{1}{2}}dx.$ $\qquad -\frac{1}{3}x^2(a-x^2)^{\frac{3}{2}} - \frac{2}{15}(a-x^2)^{\frac{5}{2}} + C.$
Assume $u = x^2$.

8. $\int x\cos x\,dx.$ $\qquad x\sin x + \cos x + C.$

185. Formulas of Reduction. We will next apply the *formula for integration by parts* to the *binomial differential*,

$$x^m(a+bx^n)^p dx,^*$$

in which p is any fraction, but m and n are whole numbers, and n is positive.

I. Let $dv = (a+bx^n)^p x^{n-1}dx$, and $u = x^{m-n+1}$;

then $\qquad v = \dfrac{(a+bx^n)^{p+1}}{nb(p+1)}$, and $du = (m-n+1)x^{m-n}dx.$

Hence, by the formula, we have

$$\int x^m(a+bx^n)^p dx = \frac{x^{m-n+1}(a+bx^n)^{p+1}}{nb(p+1)}$$
$$-\frac{m-n+1}{nb(p+1)}\int x^{m-n}(a+bx^n)^{p+1}dx. \tag{1}$$

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* That any binomial differential can be reduced to one of this form is evident. For, let the given differential be $(x^{-\frac{1}{2}} - x^{\frac{1}{3}})^{\frac{2}{3}}x^{-\frac{1}{2}}dx$; then, by multiplying the first factor and dividing the second by $(x^{\frac{1}{2}})^{\frac{2}{3}}$, we obtain $(1-x^{\frac{5}{6}})^{\frac{2}{3}}x^{-\frac{5}{6}}dx$. Putting $x = z^6$, we have $6\,(1-z^5)^{\frac{2}{3}}dz$, which is of the form required.

The formulas of reduction are true for fractional and negative values of m and n, but they are not generally useful in leading to known forms, unless m and n are whole numbers, and n positive.

Now
$$\int x^{m-n}(a+bx^n)^{p+1}dx=\int x^{m-n}(a+bx^n)^p(a+bx^n)dx$$
$$=a\int x^{m-n}(a+bx^n)^p dx+b\int x^m(a+bx^n)^p dx.$$

Substituting this value in (1), we have
$$\int x^m(a+bx^n)^p dx=\frac{x^{m-n+1}(a+bx^n)^{p+1}}{nb(p+1)}$$
$$-\frac{(m-n+1)a}{nb(p+1)}\int x^{m-n}(a+bx^n)^p dx$$
$$-\frac{m-n+1}{n(p+1)}\int x^m(a+bx^n)^p dx.$$

Transposing the last term to the first member, and solving for $\int x^m(a+bx^n)^p dx$, we obtain
$$\int x^m(a+bx^n)^p dx=\frac{x^{m-n+1}(a+bx^n)^{p+1}}{b(np+m+1)}$$
$$-\frac{a(m-n+1)}{b(np+m+1)}\int x^{m-n}(a+bx^n)^p dx. \qquad \text{(A)}$$

By formula (A) the integration of $x^m(a+bx^n)^p dx$ is made to depend upon that of another differential of the same form, in which m is diminished by n. By a repetition of its use, m may be diminished by any multiple of n.

Formula (A) evidently fails when $np+m+1=0$; but in that case $\frac{m+1}{n}+p=0$; hence the method of integration by rationalization (§ 183) is applicable, and the formula is not needed.

II. It is evident that
$$\int x^m(a+bx^n)^p dx=\int x^m(a+bx^n)^{p-1}(a+bx^n)dx$$
$$=a\int x^m(a+bx^n)^{p-1}dx$$
$$+b\int x^{m+n}(a+bx^n)^{p-1}dx. \qquad (2)$$

Applying formula (A) to the last term of (2), by substituting in the formula $m+n$ for m, and $p-1$ for p, we obtain

$$b\int x^{m+n}(a+bx^n)^{p-1}dx = \frac{x^{m+1}(a+bx^n)^p}{np+m+1} - \frac{a(m+1)}{np+m+1}\int x^m(a+bx^n)^{p-1}dx.$$

Substituting this in (2), and uniting similar terms, we have

$$\int x^m(a+bx^n)^p dx = \frac{x^{m+1}(a+bx^n)^p}{np+m+1} + \frac{anp}{np+m+1}\int x^m(a+bx^n)^{p-1}dx. \qquad \text{(B)}$$

Each application of formula (B) diminishes p, the exponent of the binomial, by unity. It fails in the same case that (A) does.

III. When m and p are negative, we need formulas to increase rather than to diminish them. To obtain these, we reverse formulas (A) and (B).

Solving (A) for $\int x^{m-n}(a+bx^n)^p dx$, and substituting $m+n$ for m, we obtain

$$\int x^m(a+bx^n)^p dx = \frac{x^{m+1}(a+bx^n)^{p+1}}{a(m+1)} - \frac{b(np+n+m+1)}{a(m+1)}\int x^{m+n}(a+bx^n)^p dx. \qquad \text{(C)}$$

Formula (C) enables us to increase m by n at each application. It fails when $m+1=0$; but in that case the differential can be rationalized (§ 183).

IV. Solving (B) for $\int x^m(a+bx^n)^{p-1}dx$, and substituting $p+1$ for p, we obtain

$$\int x^m(a+bx^n)^p dx = -\frac{x^{m+1}(a+bx^n)^{p+1}}{an(p+1)} + \frac{np+n+m+1}{an(p+1)}\int x^m(a+bx^n)^{p+1}dx. \qquad \text{(D)}$$

Formula (D) enables us to increase p by unity at each application.

The mode of applying formulas (A), (B), (C), and (D) will be illustrated by a few examples.

EXAMPLES.

1. Find $\int \frac{x^4\,dx}{(a^2 - x^2)^{\frac{1}{2}}}$.

Here $m = 4$, $n = 2$, $p = -\frac{1}{2}$, $a = a^2$, and $b = -1$. Hence, by applying formula (A) twice in succession, this integral will evidently be made to depend upon

$$\int \frac{dx}{\sqrt{a^2 - x^2}}.$$

Substituting these values of m, n, etc., in formula (A), we obtain

$$\int x^4(a^2 - x^2)^{-\frac{1}{2}}dx = -\tfrac{1}{4}x^3(a^2 - x^2)^{\frac{1}{2}} + \tfrac{3}{4}a^2\int x^2(a^2 - x^2)^{-\frac{1}{2}}dx. \qquad (1)$$

In like manner, we obtain

$$\int x^2(a^2 - x^2)^{-\frac{1}{2}}\,dx = -\tfrac{1}{2}x(a^2 - x^2)^{\frac{1}{2}} + \tfrac{1}{2}a^2\int (a^2 - x^2)^{-\frac{1}{2}}dx. \qquad (2)$$

Now, $$\int (a^2 - x^2)^{-\frac{1}{2}}dx = \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C. \qquad (3)$$

From (1), (2), and (3), we obtain

$$\int \frac{x^4\,dx}{\sqrt{a^2 - x^2}} = -(\tfrac{1}{4}x^3 + \tfrac{3}{8}a^2x)\sqrt{a^2 - x^2} + \tfrac{3}{8}a^4\sin^{-1}\frac{x}{a} + C.$$

2. $\int \frac{x^2\,dx}{\sqrt{a^2 - x^2}}$. *Ans.* $-\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C.$

3. $\int \frac{x^3\,dx}{\sqrt{a^2-x^2}}$. $\qquad -\frac{1}{3}(x^2+2a^2)(a^2-x^2)^{\frac{1}{2}}+C.$

By formula (A), $\int \frac{x^m\,dx}{\sqrt{a^2-x^2}}$ is made to depend on what known form when m is positive and even? On what known form when m is positive and odd?

4. $\int \frac{x^2\,dx}{\sqrt{a^2+x^2}}$. $\qquad \frac{x}{2}\sqrt{a^2+x^2}-\frac{1}{2}a^2\log(x+\sqrt{a^2+x^2})+C.$

5. $\int \frac{x^3\,dx}{\sqrt{a^2+x^2}}$. $\qquad \frac{1}{3}(x^2-2a^2)\sqrt{a^2+x^2}+C$

By formula (A), $\int \frac{x^m\,dx}{\sqrt{a^2+x^2}}$ is made to depend on what known form when m is positive and odd? On what known form when m is positive and even?

6. $\int \frac{x^5\,dx}{\sqrt{1-x^2}}$. $\qquad -\left(\frac{x^4}{5}+\frac{4x^2}{5\cdot 3}+\frac{4\cdot 2}{5\cdot 3}\right)\sqrt{1-x^2}+C.$

7. $\int \frac{x^6\,dx}{\sqrt{a^2-x^2}}$.

$$-\left(\frac{x^5}{6}+\frac{5a^2x^3}{6\cdot 4}+\frac{5\cdot 3a^4x}{6\cdot 4\cdot 2}\right)\sqrt{a^2-x^2}+\frac{5\cdot 3a^6}{6\cdot 4\cdot 2}\sin^{-1}\frac{x}{a}+C.$$

8. $\int (a^2-x^2)^{\frac{1}{2}}dx$. $\qquad \frac{1}{2}x(a^2-x^2)^{\frac{1}{2}}+\frac{1}{2}a^2\sin^{-1}\frac{x}{a}+C.$

Apply formula (B) once.

9. $\int x^2(1-x^2)^{\frac{1}{2}}dx$. $\qquad \frac{1}{8}x(2x^2-1)(1-x^2)^{\frac{1}{2}}+\frac{1}{8}\sin^{-1}x+C.$

10. $\int x^4(1-x^2)^{\frac{1}{2}}dx$.

$$\frac{1}{2}\left(\frac{1}{3}x^5-\frac{1}{12}x^3-\frac{1}{8}x\right)(1-x^2)^{\frac{1}{2}}+\frac{1}{16}\sin^{-1}x+C.$$

11. $\int \frac{dx}{x^2(a^2-x^2)^{\frac{1}{2}}}$. $\qquad -\frac{\sqrt{a^2-x^2}}{a^2x}+C.$

12. $\int \frac{dx}{x^3(a^2-x^2)^{\frac{3}{2}}}$. $\quad -\frac{\sqrt{a^2-x^2}}{2a^2x^2}+\frac{1}{2a^3}\log\frac{x}{\sqrt{a^2-x^2}+a}+C.$

For $\int \frac{dx}{x\sqrt{a^2-x^2}}$ see § 183, Ex. 5.

13. $\int \frac{(a^2-y^2)^{\frac{1}{2}}dy}{y}$. $\quad a\log\frac{y}{a+(a^2-y^2)^{\frac{1}{2}}}+(a^2-y^2)^{\frac{1}{2}}+C.$

Apply formula (B).

14. $\int (a^2+x^2)^{\frac{1}{2}}dx$. $\quad \frac{1}{2}x\sqrt{a^2+x^2}+\frac{1}{2}a^2\log(x+\sqrt{a^2+x^2})+C.$

15. $\int \frac{dx}{(a^2+x^2)^2}$. $\quad \frac{x}{2a^2(a^2+x^2)}+\frac{1}{2a^3}\tan^{-1}\frac{x}{a}+C.$

Apply formula (D).

16. $\int (1-x^2)^{\frac{3}{2}}dx$. $\quad \frac{1}{4}x(1-x^2)^{\frac{3}{2}}+\frac{3}{8}x(1-x^2)^{\frac{1}{2}}+\frac{3}{8}\sin^{-1}x+C.$

17. $\int \frac{x\,dx}{\sqrt{2ax-x^2}}$. $\quad -(2ax-x^2)^{\frac{1}{2}}+a\,\text{vers}^{-1}\frac{x}{a}+C.$

Write $\int \frac{x\,dx}{\sqrt{2ax-x^2}}$ in the form $\int x^{\frac{1}{2}}(2a-x)^{-\frac{1}{2}}dx$, and apply formula (A) once.

18. $\int \frac{x^2dx}{\sqrt{2ax-x^2}}$. $\quad -\frac{x+3a}{2}\sqrt{2ax-x^2}+\frac{3}{2}a^2\text{vers}^{-1}\frac{x}{a}+C.$

19. $\int \frac{x^3dx}{\sqrt{2ax-x^2}}$.

$$-\frac{2x^2+5a(x+3a)}{6}\sqrt{2ax-x^2}+\frac{5}{2}a^3\text{vers}^{-1}\frac{x}{a}+C.$$

20. $\int \frac{dx}{(a^2+x^2)^3}$.

$$\frac{x}{4a^2(a^2+x^2)^2}+\frac{3x}{8a^4(a^2+x^2)}+\frac{3}{8a^5}\tan^{-1}\frac{x}{a}+C.$$

21. $\int \frac{x^4dx}{(1-x^2)^{\frac{3}{2}}}$. $\quad \frac{x(3-x^2)}{2(1-x^2)^{\frac{1}{2}}}-\frac{3}{2}\sin^{-1}x+C.$

186. *To integrate the logarithmic differential* $\phi(\mathrm{x})\,(log\,\mathrm{x})^{\mathrm{n}}\mathrm{dx}$, *in which* $\phi(\mathrm{x})$ *is an algebraic function, and* n *is a positive whole number.*

Let $x^2(\log x)^2 dx$ be the function.

Assume $dv = x^2 dx$, and $u = (\log x)^2$;

then $du = 2\log x\frac{dx}{x}$, and $v = \frac{1}{3}x^3$.

Hence, by the formula for integration by parts, we have

$$\int x^2(\log x)^2 dx = \tfrac{1}{3}x^3(\log x)^2 - \tfrac{2}{3}\int x^2 \log x\, dx. \qquad (1)$$

Applying the formula to the last term of (1), we have

$$\int x^2 \log x\, dx = \tfrac{1}{3}x^3 \log x - \int \tfrac{1}{3}x^2 dx. \qquad (2)$$

From (1) and (2), we obtain

$$\int x^2(\log x)^2 dx = \tfrac{1}{3}x^3\left[(\log x)^2 - \tfrac{2}{3}\log x + \tfrac{2}{9}\right] + C.$$

Hence, to integrate $\phi(x)(\log x)^n dx$, we assume $dv = \phi(x)dx$, and, by successive applications of the formula for integration by parts, reduce the exponent of $\log x$ to zero. In this way the integration of the logarithmic differential is reduced to the integration of algebraic differentials.

EXAMPLES.

1. Find $\int x^3(\log x)^2 dx$. *Ans.* $\frac{1}{4}x^4\left[(\log x)^2 - \frac{1}{2}\log x + \frac{1}{8}\right] + C.$

2. $\int \frac{\log x\, dx}{(1+x)^2}$. $\quad \frac{x}{1+x}\log x - \log(1+x) + C.$

3. $\int x^n(\log x)^2 dx$. $\quad \frac{x^{n+1}}{n+1}\left[(\log x)^2 - \frac{2}{n+1}\log x + \frac{2}{(n+1)^2}\right] + C.$

4. $\int \frac{(\log x)^2}{x^{\frac{5}{2}}} dx$. $\quad -\frac{2}{3\sqrt{x^3}}\left[(\log x)^2 + \frac{4}{3}\log x + \frac{8}{9}\right] + C.$

187. *To integrate the exponential differential,* $x^n a^{mx} dx$, *when* n *is a positive integer.*

Assume $dv = a^{mx}dx$, and $u = x^n$;

then $du = nx^{n-1}dx$, and $v = \dfrac{a^{mx}}{m\log a}$.

Hence, by the formula for integration by parts, we have

$$\int x^n a^{mx} dx = \frac{x^n a^{mx}}{m\log a} - \frac{n}{m\log a}\int x^{n-1}a^{mx}dx.$$

By successive applications of this formula, the exponent of x is reduced to zero, and the proposed integral is made to depend upon the known form

$$\int a^{mx}dx.$$

EXAMPLES.

1. Find $\int x^2 e^{ax} dx$.

Assume $dv = e^{ax}dx$, and $u = x^2$;

then $$\int x^2 e^{ax} dx = \frac{1}{a}e^{ax}x^2 - \frac{2}{a}\int x e^{ax} dx.$$

Again, $$\int x e^{ax} dx = \frac{1}{a}e^{ax}x - \frac{1}{a}\int e^{ax}dx.$$

Hence $$\int x^2 e^{ax} dx = \frac{e^{ax}}{a}\left(x^2 - \frac{2x}{a} + \frac{2}{a^2}\right) + C.$$

2. $\int x^3 e^{ax} dx.$ $\qquad \dfrac{e^{ax}}{a}\left(x^3 - \dfrac{3}{a}x^2 + \dfrac{3\cdot 2}{a^2}x - \dfrac{3\cdot 2\cdot 1}{a^3}\right) + C.$

3. $\int x^3 a^x dx.$ $\qquad \dfrac{a^x}{\log a}\left[x^3 - \dfrac{3\cdot x^2}{\log a} + \dfrac{3\cdot 2\cdot x}{(\log a)^2} - \dfrac{3\cdot 2\cdot 1}{(\log a)^3}\right] + C.$

4. Write out the integral of $e^x x^4 dx$, according to the law of the integrals in Examples 2 and 3.

188. *To integrate the trigonometric differential* $\sin^m x \cos^n x dx$.

Let $\sin x = z$;

then $\sin^m x = z^m$, $\cos^n x = (1 - z^2)^{\frac{n}{2}}$, and $\qquad dx=(1-z^2)^{-\frac{1}{2}}dz.$

Hence,
$$\int \sin^m x \cos^n x\,dx=\int z^m(1-z^2)^{\frac{n-1}{2}}dz. \qquad (1)$$

Or, letting $\cos x = z$, we obtain
$$\int \sin^m x \cos^n x\,dx=\int -z^n(1-z^2)^{\frac{m-1}{2}}dz. \qquad (2)$$

Hence, whenever the binomial differential in (1) or (2) can be integrated, the given differential can be.

This method of integrating $\sin^m x \cos^n x\,dx$ is used in those cases to which the shorter methods of § 63 are not applicable.

Examples.

1. Find $\int \sin^6 x\,dx$.

Put $\qquad \sin x = z;$

then $\qquad dx=(1-z^2)^{-\frac{1}{2}}dz,$

and
$$\begin{aligned}\int \sin^6 x\,dx &= \int z^6(1-z^2)^{-\frac{1}{2}}dz\\ &= -\left(\frac{z^5}{6}+\frac{5z^3}{6\cdot 4}+\frac{5\cdot 3z}{6\cdot 4\cdot 2}\right)(1-z^2)^{\frac{1}{2}}\\ &\quad+\frac{5\cdot 3}{6\cdot 4\cdot 2}\sin^{-1}z+C \qquad \text{§ 185, Ex. 7.}\\ &= -\frac{\cos x}{6}\left(\sin^5 x+\tfrac{5}{4}\sin^3 x+\frac{5\cdot 3}{4\cdot 2}\sin x\right)\\ &\quad+\frac{5\cdot 3}{6\cdot 4\cdot 2}x+C.\end{aligned}$$

2. $\int \sin^4 x\,dx.$ *Ans.* $-\frac{1}{4}\cos x\,(\sin^3 x+\frac{3}{2}\sin x)+\frac{3}{8}x+C.$

3. $\int \cos^4 x\,dx.$ $\frac{1}{4}\sin x\,(\cos^3 x+\frac{3}{2}\cos x)+\frac{3}{8}x+C.$

4. $\int \sin^2 x\cos^2 x\,dx.$ $\frac{1}{4}\sin^3 x\cos x-\frac{1}{8}\sin x\cos x+\frac{1}{8}x+C,$
or $\frac{1}{8}(x-\frac{1}{4}\sin 4x)+C.$

Let $\sin x = z$, and, for $\int z^2(1-z^2)^{\frac{1}{2}}dz$, see §185, Ex. 9.

The second form of the integral is obtained from the first by use of the relations, $2\sin x\cos x = \sin 2x$, and $2\sin^2 x = 1 - \cos 2x$. By a similar transformation, any differential or integral expressed in powers of $\sin x$ and $\cos x$ may be found in terms of the sines and cosines of multiples of x.

5. $\int \sin^2 x \cos^4 x dx.$ $\qquad \frac{\sin x}{2}\left(\frac{\cos^3 x}{12} - \frac{\cos^5 x}{3} + \frac{\cos x}{8}\right) + \frac{x}{16} + C.$

6. $\int \frac{dx}{\sin x \cos^3 x}.$ $\qquad \frac{1}{2\cos^2 x} + \log\tan x + C.$

7. $\int \frac{dx}{\sin^3 x}.$ $\qquad -\frac{\cos x}{2\sin^2 x} + \frac{1}{2}\log\tan\frac{x}{2} + C.$

8. $\int \frac{\cos^4 x dx}{\sin x}.$ $\qquad \frac{1}{3}\cos^3 x + \cos x + \log\tan\frac{x}{2} + C.$

189. *To integrate* $x^n \sin(ax)dx$, *and* $x^n \cos(ax)dx$.

Assume $u = x^n$, and apply the formula for integration by parts. Each application of the formula will evidently diminish n by unity; hence, when n is a positive integer, the integral can be made to depend on the known form

$$\int \cos(ax)\,dx \text{ or } \int \sin(ax)dx.$$

190. *To integrate* $e^{ax}\sin^n x dx$, *and* $e^{ax}\cos^n x dx$.

Put $dv = e^{ax}dx$, and $u = \sin^n x$;

then $\qquad v = \frac{1}{a}e^{ax},$

and $\qquad du = n\sin^{n-1}x\cos x dx.$

$$\therefore \int e^{ax}\sin^n x dx = \frac{1}{a}e^{ax}\sin^n x$$
$$-\frac{n}{a}\int e^{ax}\sin^{n-1}x\cos x dx. \qquad (1)$$

Again, put $dv = e^{ax}dx$, and $u = \sin^{n-1}x\cos x$;

then $\qquad v = \frac{1}{a}e^{ax},$

and $$du = (n-1)\sin^{n-2}x\cos^2 x\,dx - \sin^n x\,dx$$
$$= (n-1)\sin^{n-2}x\,dx - n\sin^n x\,dx.$$
[Since $\cos^2 x = 1 - \sin^2 x$.]

$$\therefore \int e^{ax}\sin^{n-1}x\cos x\,dx = \frac{1}{a}e^{ax}\sin^{n-1}x\cos x$$
$$-\frac{n-1}{a}\int e^{ax}\sin^{n-2}x\,dx + \frac{n}{a}\int e^{ax}\sin^n x\,dx.$$

Substituting this result in (1), and solving for $\int e^{ax}\sin^n x\,dx$, we obtain,

$$\int e^{ax}\sin^n x\,dx = \frac{e^{ax}\sin^{n-1}x(a\sin x - n\cos x)}{n^2+a^2}$$
$$+\frac{n(n-1)}{n^2+a^2}\int e^{ax}\sin^{n-2}x\,dx. \qquad (2)$$

By repeating this process or the application of this formula, n is reduced to zero or unity; and the integral is made to depend upon the known form $\int e^{ax}dx$ or the form $\int e^{ax}\sin x\,dx$. The value of the latter form is obtained directly from (2) by making $n = 1$.

In like manner $\int e^{ax}\cos x\,dx$ can be obtained.

EXAMPLES.

1. Find $\int x^2\cos x\,dx$.

Here $u = x^2$, $dv = \cos x\,dx$,

$v = \sin x$, and $du = 2x\,dx$.

$$\therefore \int x^2\cos x\,dx = x^2\sin x - 2\int x\sin x\,dx$$
$$= x^2\sin x + 2x\cos x - 2\int\cos x\,dx$$
$$= x^2\sin x + 2x\cos x - 2\sin x + C.$$

2. $\int x^3\sin x\,dx$. $\quad -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C$.

3. $\int e^{ax}\sin x\,dx.$ $\qquad \dfrac{e^{ax}}{a^2+1}(a\sin x-\cos x)+C.$

4. $\int e^{x}\sin^3 x\,dx.$ $\qquad \dfrac{e^{x}}{10}(\sin^3 x+3\cos^3 x+3\sin x-6\cos x)+C.$

191. *To integrate* $\dfrac{dx}{a+b\cos x}$.

$$\int\frac{dx}{a+b\cos x}=\int\frac{dx}{a\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)+b\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}$$

$$=\int\frac{dx}{(a+b)\cos^2\frac{x}{2}+(a-b)\sin^2\frac{x}{2}}$$

$$=\int\frac{\sec^2\frac{x}{2}dx}{(a+b)+(a-b)\tan^2\frac{x}{2}}$$

$$=2\int\frac{d\left(\tan\frac{x}{2}\right)}{(a+b)+(a-b)\tan^2\frac{x}{2}},$$

which is readily reduced to the known form,

$$\int\frac{dx}{c^2+x^2}\text{ or }\int\frac{dx}{c^2-x^2},\text{ according as } a>\text{ or }<b.$$

In like manner $\int\dfrac{dx}{a+b\sin x}$ can be found.

192. *To integrate the anti-trigonometric differentials,*

$f(x)\sin^{-1}x\,dx,\ f(x)\cos^{-1}x\,dx,\ f(x)\tan^{-1}x\,dx,$ *etc.*,

in which f (x) *is an algebraic function.*

Assume $dv=f(x)\,dx$, and apply the formula for integrating by parts. One application of the formula will evidently make the integral depend on an algebraic form.

EXAMPLES.

1. Find $\int \frac{x^2dx}{1+x^2}\tan^{-1}x$.

Here $dv=\frac{x^2dx}{1+x^2}=dx-\frac{dx}{1+x^2}$, $u=\tan^{-1}x$,

$$v=x-\tan^{-1}x, \text{ and } du=\frac{dx}{1+x^2}.$$

$$\begin{aligned}\therefore \int \frac{x^2dx}{1+x^2}\tan^{-1}x &= x\tan^{-1}x-(\tan^{-1}x)^2\\ &\quad -\int\frac{x\,dx}{1+x^2}+\int\frac{\tan^{-1}x\,dx}{1+x^2}\\ &= x\tan^{-1}x-(\tan^{-1}x)^2\\ &\quad -\tfrac{1}{2}\log(1+x^2)+\tfrac{1}{2}(\tan^{-1}x)^2+C\\ &=\tan^{-1}x\,(x-\tfrac{1}{2}\tan^{-1}x)-\log\sqrt{1+x^2}+C.\end{aligned}$$

2. $\int x\cos^{-1}x\,dx.$ $\qquad \frac{1}{2}x^2\cos^{-1}x-\frac{1}{4}x(1-x^2)^{\frac{1}{2}}+\frac{1}{4}\sin^{-1}x+C.$

3. $\int x^2\sin^{-1}x\,dx.$ $\qquad \frac{1}{3}x^3\sin^{-1}x+\frac{1}{9}(x^2+2)\sqrt{1-x^2}+C.$

193. Integration by Series. When we cannot, by any of the preceding methods, integrate a given differential exactly; or, when the integral obtained by them is of a complicated form, we can develop the given differential in a series, and integrate its terms separately. Moreover, integration by series furnishes a simple method of developing a function, when we know the development of its derivative. For examples of this method of developing functions, see §§ 107, 108.

EXAMPLES.

1. Find $\int x^{\frac{1}{2}}(1-x^2)^{\frac{1}{2}}dx$.

$$(1-x^2)^{\frac{1}{2}}=1-\tfrac{1}{2}x^2-\tfrac{1}{8}x^4-\tfrac{1}{16}x^6-\cdots;$$

$$\therefore \int x^{\frac{1}{2}}(1-x^2)^{\frac{1}{2}}dx = \int x^{\frac{1}{2}}(1-\tfrac{1}{2}x^2-\tfrac{1}{8}x^4-\tfrac{1}{16}x^6-\cdots)dx$$
$$= \tfrac{2}{3}x^{\frac{3}{2}}-\tfrac{1}{7}x^{\frac{7}{2}}-\tfrac{1}{44}x^{\frac{11}{2}}-\tfrac{1}{120}x^{\frac{15}{2}}-\cdots+C,$$

which is the required integral for $x<1$ and >-1.

2. $\int x^2(1-x^2)^{\frac{1}{2}}dx.$ $\quad \tfrac{1}{3}x^3-\tfrac{1}{10}x^5-\tfrac{1}{56}x^7-\tfrac{1}{144}x^9-\cdots+C.$

3. Prove that $\log(a+x)=\log a+\dfrac{x}{a}-\dfrac{x^2}{2a^2}+\dfrac{x^3}{3a^3}-\dfrac{x^4}{4a^4}+\cdots,$

by first integrating $\dfrac{dx}{a+x}$ directly, and then by series.

4. Develop $\log(x+\sqrt{1+x^2})$ by integrating $\dfrac{dx}{\sqrt{1+x^2}}$.

Ans. $\log(x+\sqrt{1+x^2})=x-\dfrac{1}{2}\,\dfrac{x^3}{3}+\dfrac{1\cdot 3}{2\cdot 4}\,\dfrac{x^5}{5}-\dfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\,\dfrac{x^7}{7}+\cdots$

5. Prove that

$$\int\frac{dx}{(1+x^4)^{\frac{1}{2}}}=x-\frac{1}{2}\,\frac{x^5}{5}+\frac{1\cdot 3}{2\cdot 4}\,\frac{x^9}{9}-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\,\frac{x^{13}}{13}+\cdots+C.$$

194. While we can differentiate any given integral, we can exactly integrate but a small number of differentials. One great reason for this seeming difference in the perfection of the two branches of the Calculus is that the integral is often a higher or more complex function than its differential. Thus, the differentials of $\log x$, $\sin^{-1}x$, $\tan^{-1}x$, etc., are algebraic functions. Moreover, it is evident that certain forms of differentials do not arise from the differentiation of any known functions. Hence, to obtain the exact integrals of many differentials, new and higher functions must be invented and studied. The integrals of these differentials, as obtained by series, are the developments of these, as yet, unknown functions.

CHAPTER XVI.

LENGTHS AND AREAS OF PLANE CURVES, AREAS OF SURFACES OF REVOLUTION, VOLUMES OF SOLIDS.

195. Examples in Rectification of Plane Curves. For formulas, see § 65.

1. Find the length of the parabola $y^2 = 2px$.

Here $$s = \int\left(1+\frac{dx^2}{dy^2}\right)^{\frac{1}{2}}dy = \int\left(1+\frac{y^2}{p^2}\right)^{\frac{1}{2}}dy = \frac{1}{p}\int(p^2+y^2)^{\frac{1}{2}}dy;$$

$$\therefore s = \frac{y\sqrt{p^2+y^2}}{2p} + \frac{p}{2}\log(y+\sqrt{p^2+y^2}) + C. \quad \text{§ 185, Ex. 14.}$$

If s be measured from the origin, $s = 0$ when $y = 0$, and

$$C = -\tfrac{1}{2}p\log p.$$

$$\therefore s = \frac{y}{2p}\sqrt{p^2+y^2} + \frac{p}{2}\log\frac{y+\sqrt{p^2+y^2}}{p}.$$

2. Rectify the circle $x^2 + y^2 = r^2$.

We use L to represent the entire length of any closed curve.

Here $$L = 4\int_0^r\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}dx = 4r\int_0^r\frac{dx}{\sqrt{r^2-x^2}} = 2\pi r.$$

For the value of π, see § 107.

3. Rectify the ellipse $y^2 = (1-e^2)(a^2-x^2)$.

Here $$\frac{dy}{dx} = -(1-e^2)\frac{x}{y} = -\frac{x\sqrt{1-e^2}}{\sqrt{a^2-x^2}};$$

$$\therefore L = 4\int_0^a\left(\frac{a^2-e^2x^2}{a^2-x^2}\right)^{\frac{1}{2}}dx = 4\int_0^a\frac{dx}{\sqrt{a^2-x^2}}(a^2-e^2x^2)^{\frac{1}{2}}$$

$$= 4\int_0^a\frac{dx}{\sqrt{a^2-x^2}}\left(a - \frac{e^2x^2}{2a} - \frac{e^4x^4}{2\cdot4a^3} - \frac{3e^6x^6}{2\cdot4\cdot6a^5} - \cdots\right)$$

$$= 4a\int_0^a \frac{dx}{\sqrt{a^2-x^2}} - \frac{2e^2}{a}\int_0^a \frac{x^2\,dx}{\sqrt{a^2-x^2}} - \frac{e^4}{2a^3}\int_0^a \frac{x^4\,dx}{\sqrt{a^2-x^2}}$$

$$- \frac{3e^6}{2\cdot 6a^5}\int_0^a \frac{x^6\,dx}{\sqrt{a^2-x^2}} - \cdots$$

For the indefinite integrals of the last three terms, see § 185, Examples 1, 2, and 7. Finding the definite integrals between the given limits, and adding the results, we have

$$L = 2\pi a\left(1 - \frac{e^2}{2^2} - \frac{3e^4}{2^2\cdot 4^2} - \frac{3^2\cdot 5e^6}{2^2\cdot 4^2\cdot 6^2} - \cdots\right).$$

4. Rectify the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. *Ans.* $6a$.

5. Rectify the tractrix.

The characteristic property of the tractrix is that the length of its tangent PT is constant. Denote this constant length by a; and let O be the origin, OA being the tangent at A; then, if PM $= ds$, $-$PN $= dy$, NM $= dx$, ET $= \sqrt{a^2 - y^2}$.

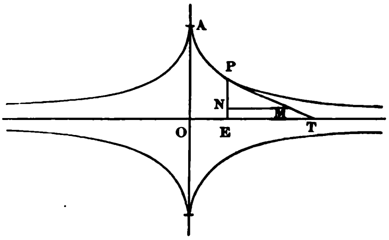

Fig. 54.

$$\text{Hence } \frac{ds}{dy} = -\frac{\text{PM}}{\text{PN}} = -\frac{a}{y}; \text{ also, } \frac{dy}{dx} = -\frac{y}{\sqrt{a^2-y^2}}. \qquad (1)$$

$$\therefore s = -a\int\frac{dy}{y} = a\log\left(\frac{a}{y}\right), \text{ if } s \text{ be measured from A.}$$

In this example we have found the length of a curve without knowing its equation. For the length of the catenary obtained in a similar way, see § 76, Ex. 8.

196. *To rectify a curve given by its polar equation.*

Since $ds^2 = \rho^2 d\theta^2 + d\rho^2$, § 136 (3).

$$s = \int\left(\rho^2 + \frac{d\rho^2}{d\theta^2}\right)^{\frac{1}{2}} d\theta, \text{ or } \int\left(1 + \rho^2\frac{d\theta^2}{d\rho^2}\right)^{\frac{1}{2}} d\rho.$$

EXAMPLES.

1. Rectify the spiral of Archimedes, $\rho = a\theta$.

Here $\dfrac{d\theta}{d\rho} = \dfrac{1}{a}$; $\therefore s = \dfrac{1}{a}\int (a^2 + \rho^2)^{\frac{1}{2}} d\rho$.

Hence, if s be measured from the pole, we have

$$s = \frac{\rho(a^2+\rho^2)^{\frac{1}{2}}}{2a} + \frac{a}{2}\log\frac{\rho + \sqrt{a^2+\rho^2}}{a}. \quad \S\,185,\ \text{Ex. } 14.$$

This is equal to the arc of the parabola, $y^2 = 2ax$, intercepted between the vertex and the point whose ordinate equals ρ (§ 195, Ex. 1).

2. Rectify the logarithmic spiral $\rho = a^\theta$.

Here $s = \int (1+m^2)^{\frac{1}{2}} d\rho = (1+m^2)^{\frac{1}{2}}\rho$,

in which m is the modulus of the system of logarithms whose base is a, and s is measured from the pole.

3. Construct and rectify the cardioid $\rho = a(1+\cos\theta)$.

$$L = 2\int_0^\pi [a^2(1+\cos\theta)^2 + a^2\sin^2\theta]^{\frac{1}{2}} d\theta$$

$$= 8a\int_0^\pi \cos\frac{\theta}{2}\frac{d\theta}{2} = 8a;$$

since $2(1+\cos\theta) = 4\cos^2\dfrac{\theta}{2}$.

197. Examples in Quadrature of Plane Curves. The *quadrature* of a figure or surface is the finding of its area. For formulas, see § 66.

1. Find the area of the circle $x^2 + y^2 = r^2$.

Here $\text{area} = 4\int_0^r (r^2 - x^2)^{\frac{1}{2}} dx$

$$= 4\left[\frac{x(r^2-x^2)^{\frac{1}{2}}}{2} + \frac{r^2}{2}\sin^{-1}\frac{x}{r}\right]_0^r = \pi r^2. \quad \S\,185,\ \text{Ex. } 8.$$

For the segment between the lines $x = a$ and $x = b$, we have

$$\text{area} = 2\int_a^b (r^2 - x^2)^{\frac{1}{2}} dx$$

$$= b\sqrt{r^2 - b^2} + r^2 \sin^{-1}\frac{b}{r} - a\sqrt{r^2 - a^2} - r^2 \sin^{-1}\frac{a}{r}.$$

2. Find the area of one branch of the cycloid,

$$x = r \operatorname{vers}^{-1}\frac{y}{r} - \sqrt{2ry - y^2}.$$

$$\text{Area} = 2\int_0^{2r} \frac{y^2 dy}{\sqrt{2ry - y^2}} = 3\pi r^2. \qquad \text{§ 185, Ex. 18.}$$

Hence the area of one branch is three times that of the generating circle.

3. Find the area of the tractrix.

Here $$\frac{dy}{dx} = -\frac{y}{\sqrt{a^2 - y^2}}, \text{or } dx = -\frac{\sqrt{a^2 - y^2}}{y} dy. \quad \text{§195, Ex. 5.}$$

$$\therefore z = \int y\, dx = -\int \sqrt{a^2 - y^2}\, dy;$$

$$\therefore \text{area} = -4\int_a^0 \sqrt{a^2 - y^2}\, dy = \pi a^2. \qquad \text{§ 185, Ex. 8.}$$

Hence the whole area enclosed by the curve is equal to the area of a circle whose radius is a.

4. Find the whole area between the cissoid $y^2 = \dfrac{x^3}{2a - x}$ and its asymptote.

Ans. $3\pi a^2$.

5. Find the area between the lines $x^2 y = a^3$, $x = b$, $x = c$, and $y = 0$.

Ans. $a^3 \dfrac{b - c}{bc}$.

6. Find the area of both loops of the curve $a^4 y^2 = a^2 b^2 x^2 - b^2 x^4$.

Ans. $\frac{4}{3}ab$.

7. Find the area of one loop of the curve $a^2 y^4 = x^4(a^2 - x^2)$.

Ans. $\frac{4}{5}a^2$.

198. *To find the area of a curve given by its polar equation.*

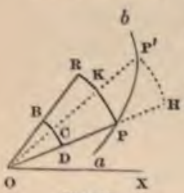

Fig. 55.

Let P be any point on the curve ab referred to the pole O and the polar axis OX. Take $\mathrm{OD} = 1$, and draw the arcs DB and PR; then, if $\mathrm{DB} = d\theta$, it is evident that the sector $\mathrm{OPR} = dA$, A representing the area traced by the radius vector.

Hence $\quad dA = \frac{1}{2}\mathrm{PR}\cdot\mathrm{OP} = \frac{1}{2}\rho^2 d\theta;$

$$\therefore A = \tfrac{1}{2}\int \rho^2 d\theta.$$

Or, let $\quad \mathrm{DC} = \Delta\theta$; then $\mathrm{KP}' = \Delta\rho$, and area $\mathrm{OPP}' = \Delta A$.

Now $\quad \mathrm{OPK} < \mathrm{OPP}' < \mathrm{OHP}'$, or $\dfrac{\rho^2}{2}\Delta\theta < \Delta A < \dfrac{(\rho + \Delta\rho)^2}{2}\Delta\theta.$

$$\therefore \frac{\rho^2}{2} < \frac{\Delta A}{\Delta\theta} < \frac{(\rho + \Delta\rho)^2}{2}; \quad \therefore \frac{dA}{d\theta} = \frac{\rho^2}{2}, \text{ or } A = \tfrac{1}{2}\int \rho^2 d\theta.$$

EXAMPLES.

1. Find the area of the first spire of the spiral of Archimedes, $\rho = a\theta$; also the area between the first spire and the second.

Here $\quad A = \frac{1}{2}\int a^2\theta^2 d\theta = \frac{1}{6}a^2\theta^3 = \frac{1}{6}\rho^2\theta.$

$C = 0$, since, if the area be estimated from the pole, $A = 0$ when $\theta = 0$.

When $\theta = 2\pi$, $\rho = r$, or the radius of the measuring circle; and $A = \frac{1}{3}\pi r^2$. Hence the area of the first spire is one-third of the area of the measuring circle.

When $\theta = 4\pi$, $\rho = 2r$, and $A = \frac{8}{3}\pi r^2$.

But, in the two revolutions, the area of the first spire has been traced twice; hence the area between the first spire and the second is $\frac{8}{3}\pi r^2 - \frac{2}{3}\pi r^2$, or twice the area of the measuring circle. The area between the second spire and the third is four times the area of the measuring circle; and so on.

2. Find the area of the curve $\rho = a\sin 3\theta$.

The curve consists of three equal loops (Fig. 47). Hence the area equals three times the area of the first loop.

Ans. $\frac{1}{4}\pi a^2$.

3. Find the area of the lemniscate $\rho^2 = a^2 \cos 2\theta$.

The integral between $\theta = 0$ and $\theta = \frac{1}{4}\pi$ is one-fourth of the whole area.

Ans. a^2.

4. Find the area of the cardioid $\rho = a(\cos\theta + 1)$.

Ans. $\frac{3}{2}\pi a^2$.

199. Examples in Quadrature of Surfaces of Revolution. For formulas, see § 68.

1. Find the area of the surface of the prolate spheroid; that is, of the surface traced by the revolution of the ellipse

$$y^2 = (1 - e^2)(a^2 - x^2)$$

about the axis of x.

$$\text{Area} = 2\int_0^a 2\pi(1 - e^2)^{\frac{1}{2}}(a^2 - x^2)^{\frac{1}{2}}\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}} dx$$

$$= 4\pi e(1 - e^2)^{\frac{1}{2}}\int_0^a \left(\frac{a^2}{e^2} - x^2\right)^{\frac{1}{2}} dx$$

$$= 4\pi e\frac{b}{a}\int_0^a \left(\frac{a^2}{e^2} - x^2\right)^{\frac{1}{2}} dx$$

$$= 4\pi e\frac{b}{a}\left[\tfrac{1}{2}x\left(\frac{a^2}{e^2} - x^2\right)^{\frac{1}{2}} + \frac{a^2}{2e^2}\sin^{-1}\frac{ex}{a}\right]_0^a \quad \text{§ 185, Ex. 8.}$$

$$= 2\pi b^2 + \frac{2\pi ab}{e}\sin^{-1}e.$$

2. Find the area of the surface of the prolate spheroid whose generatrix is $9y^2 + 4x^2 = 36$.

3. Find the area of the surface generated by the revolution of the cycloid about its base.

$$\text{Area} = 2\int_0^{2r} 2\pi y\,ds = 4\pi\int_0^{2r} y\left(1 + \frac{dx^2}{dy^2}\right)^{\frac{1}{2}} dy$$

$$= 4\pi\sqrt{2r}\int_0^{2r} y(2r - y)^{-\frac{1}{2}} dy$$

$$= 4\pi\sqrt{2r}\left[-\tfrac{2}{3}(4r + y)(2r - y)^{\frac{1}{2}}\right]_0^{2r} \quad \text{§ 179, Ex. 3.}$$

$$= \tfrac{64}{3}\pi r^2.$$

4. Find the area of the surface generated by the catenary revolving about the axis of x, between the limits 0 and b.

Here
$$S = 2\pi\int_0^b y\,ds = \pi a\int_0^b\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)ds$$
$$= \tfrac{1}{2}\pi a\int_0^b\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)^2 dx \qquad \S\,76,\ \text{Ex. }8.$$
$$= \pi\left[\frac{a^2}{4}\left(e^{\frac{2b}{a}} - e^{-\frac{2b}{a}}\right) + ab\right].$$

5. Find the area of the surface generated by the revolution of the tractrix about the axis of x. (See § 195, Ex. 5.)

Ans. $4\pi a^2$.

200. Examples in Cubature of Solids of Revolution. The *cubature* of a solid is the finding of its volume. For formulas see § 69.

1. Find the volume of the solid generated by the revolution of the cycloid about its base.

Here
$$dx = \frac{y\,dy}{\sqrt{2ry - y^2}}; \quad \therefore \pi y^2\,dx = \frac{\pi y^3\,dy}{\sqrt{2ry - y^2}}.$$
$$\therefore \text{volume} = 2\pi\int_0^{2r}\frac{y^3\,dy}{\sqrt{2ry - y^2}} = 5\pi^2 r^3; \qquad \S\,185,\ \text{Ex. }19.$$

that is, the volume is five-eighths of the circumscribed cylinder.

2. Find the inclosed volume of the solid generated by the revolution of the parabola $y^2 = 2px$ about the line $x = a$.

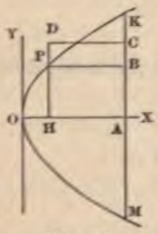

Fig. 56.

Let MK be the line $x = a$; let $b = \text{AK}[= \sqrt{2pa}]$; and let P be any point on the parabola; then $\text{OH} = x$, and $\text{HA} = a - x$. Now, if $\text{BC} = dy$, the volume generated by the revolution of BPDC about MK equals dV;

$$\therefore dV = \pi(a - x)^2 dy;$$
$$\therefore \text{volume} = 2\pi\int_0^b\left(a - \frac{y^2}{2p}\right)^2 dy = \tfrac{16}{15}\pi b a^2.$$

3. Find the volume of the solid generated by the revolution of the cissoid about its asymptote.

Ans. $2\pi^2 a^3$.

4. Find the volume of the solid generated by the revolution of the tractrix about the axis of x.

Since $y=a$ when $x=0$, and $y=0$ when $x=\infty$, and

$$dx=-\frac{\sqrt{a^2-y^2}}{y}\,dy;$$

$$\text{volume}=2\int_{x=0}^{x=\infty}\pi y^2\,dx=-2\pi\int_{y=a}^{y=0}y\sqrt{a^2-y^2}\,dy=\tfrac{2}{3}\pi a^3.$$

201. The Calculus is often of great aid in deducing the equations of curves. The equation of the catenary is obtained by its use in § 76, Ex. 8.

EXAMPLES.

1. Find the equation of the tractrix.

Here $\dfrac{dy}{dx}=-\dfrac{y}{(a^2-y^2)^{\frac{1}{2}}}$; § 195, Ex. 5.

$$\therefore x=-\int\frac{(a^2-y^2)^{\frac{1}{2}}dy}{y};$$

$$\therefore x=a\log\frac{a+(a^2-y^2)^{\frac{1}{2}}}{y}-(a^2-y^2)^{\frac{1}{2}}. \qquad \text{§ 185, Ex. 13.}$$

$C=0$, since $x=0$ when $y=a$.

2. Find the equation of the curve whose subtangent is c.

Here $y\dfrac{dx}{dy}[=\text{subt.}]=c$;

$$\therefore dx=c\frac{dy}{y};$$

$$\therefore\ x=\frac{c}{m}\log_a y+C.$$

If $c=m$, and the curve pass through the point (0, 1), we have

$$x=\log_a y.$$

3. Find the equation of the curve whose subnormal is c times the square of its abscissa. *Ans.* $y^2=\frac{2}{3}cx^3+C.$

CHAPTER XVII.

THE METHOD OF INFINITESIMALS.

202. Infinitesimals and Infinites. A quantity so small that its value cannot be expressed in terms of a finite unit, is said to be *infinitely small.*

An **Infinitesimal** is an *infinitely small* variable whose limit is zero. For example, if $y = f(x)$, Δx and Δy both become infinitesimals as $\Delta x \doteq 0$. Again, any variable, when near its limit, differs from its limit by an infinitesimal.

When we consider several related infinitesimals, we choose arbitrarily some one of them as the *principal infinitesimal*, and adopt the following definitions:

Any infinitesimal, the limit of whose ratio to the principal infinitesimal is finite, is an *infinitesimal of the first order.*

Any infinitesimal, the limit of whose ratio to the square of the principal infinitesimal is finite, is an *infinitesimal of the second order.*

Any infinitesimal, the limit of whose ratio to the nth power of the principal infinitesimal is finite, is an *infinitesimal of the* n*th order.*

Hence, if ι represent the principal infinitesimal, $v_1\iota$, $v_2\iota^2$, $v_3\iota^3$, and $v_n\iota^n$ will represent respectively any infinitesimals of the first, second, third, and nth orders, in which v_1, v_2, v_3, and v_n are variables having finite limits, from which they differ by infinitesimals. According to this notation, $\alpha = \iota$, $\alpha = v_1\iota$, and $\alpha = v_2\iota^2$ are read respectively, "$\alpha =$ the principal infinitesimal," "$\alpha =$ an infinitesimal of the first order," and "$\alpha =$ an infinitesimal of the second order."

A quantity so large that its value cannot be expressed in terms of a finite unit, is said to be *infinitely large.*

An **Infinite** is an infinitely large variable that increases without limit. Hence the reciprocals of infinitesimals are infinites, and the different orders of infinites may be represented by $w_1\iota^{-1}$, $w_2\iota^{-2}$, $w_3\iota^{-3}$, etc., in which w_1, w_2, w_3, etc., are variables having finite limits.

The symbols 0 and ∞, or $\frac{1}{0}$, represent respectively absolute zero and absolute infinity, of which there are no orders.

203. From the algebraic symbols for infinitesimals and infinites of different orders, the following principles are evident:

1. The product of any infinitesimal and an infinite of the same order is a finite quantity; thus, $v_2\iota^2 \cdot w_2\iota^{-2} = v_2w_2$.

2. The order of the product of two or more infinitesimals is the sum of the orders of the factors; thus, $v_1\iota \cdot v_2\iota^2 = v_1v_2\iota^3$.

3. The order of the quotient of any two infinitesimals is the order of the dividend minus the order of the divisor; thus,

$$\frac{v_2\iota^2}{v_1\iota} = \frac{v_2}{v_1}\iota.$$

4. If the limit of the ratio of one infinitesimal to another is zero, the former is of a *higher* order than the latter; thus,

$$\text{limit}\,\frac{v_2\iota^2}{v_1\iota} = \text{limit}\,\frac{v_2}{v_1}\iota = 0.$$

204. Geometric Illustration of Infinitesimals of Different Orders. Let CAB be a right angle inscribed in the semicircle CAB, BD a tangent at B, and AE a perpendicular to BD. From the similar triangles CAB, BAD, and AED, we have

$$\frac{\text{AD}}{\text{AB}} = \frac{\text{AB}}{\text{AC}}, \qquad (1)$$

and

$$\frac{\text{DE}}{\text{AD}} = \frac{\text{AB}}{\text{BC}}. \qquad (2)$$

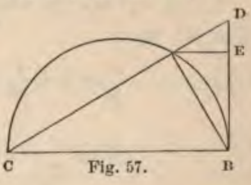

Fig. 57.

Suppose A to approach B so that AB $= \iota$. Since limit AB $= 0$, and limit AC $=$ CB, from (1) we have

$$\text{limit}\,\frac{\text{AD}}{\text{AB}} = \text{limit}\,\frac{\text{AB}}{\text{AC}} = 0;$$

hence AD is an infinitesimal of a higher order than AB (§ 203, 4).

From (2) we have

$$\text{limit}\,\frac{\text{DE}}{\text{AD}} = \text{limit}\,\frac{\text{AB}}{\text{BC}} = 0;$$

hence DE is an infinitesimal of a higher order than AD.

Thus, when $\text{AB} = \iota$, $\text{AD} = v_2\iota^2$, and $\text{DE} = v_3\iota^3$.

EXAMPLES.

1. If $a = \iota$, of what order is $\sin a$?

Since $\displaystyle \underset{a \doteq 0}{\text{limit}}\left[\frac{\sin a}{a}\right] = 1$, $\sin a = v_1\iota$ (§ 202),

and $\text{limit}\; v_1 = 1$.

2. If $a = \iota$, of what order is $\tan a$?

3. If $a = \iota$, of what order is $1 - \cos a$?

Since $\displaystyle \underset{a \doteq 0}{\text{limit}}\left[\frac{1-\cos a}{a^2}\right] = \frac{1}{2}$, $1 - \cos a = v_2\iota^2$ (§ 202),

and $\text{limit}\; v_2 = \frac{1}{2}$.

4. If $a = \iota$, show that $\sin a - a = v_3\iota^3$, and that $a - \tan a = v_3\iota^3$.

205. First Fundamental Principle of Infinitesimals.

Let $a - \beta = \epsilon$, in which ϵ is infinitely small in comparison with a or β; then

$$\frac{a}{\beta} = 1 + \frac{\epsilon}{\beta}, \text{ and limit}\,\frac{\epsilon}{\beta} = 0;$$

$$\therefore \text{limit}\,\frac{a}{\beta} = \text{limit}\left[1 + \frac{\epsilon}{\beta}\right] = 1.$$

Hence, *if the difference between two variables is infinitely small in comparison with either of them, the limit of their ratio is unity;*

and, by § 135, *either of them may be substituted for the other in any problem concerning the limit of the ratio of two variables.*

For convenience of application, this principle may be stated as follows:

In problems concerning the limit of the ratio of two variables,

All infinitesimals of the higher orders may be dropped from sums of infinitesimals of different orders.

All infinitesimals may be dropped from sums of finite quantities and infinitesimals.

All finite quantities may be dropped from sums of infinites and finite quantities.

COR. If $\alpha=\beta+\epsilon$, limit $\frac{\alpha}{\beta}$, or limit $\left[1+\frac{\epsilon}{\beta}\right]$, is unity only when ϵ is infinitely small in comparison with β.

Hence, conversely, *if the limit of the ratio of two infinitesimals is unity, their difference is infinitely small in comparison with either.*

206. Rule for Differentiation. In this chapter we shall regard the increments of variables as infinitesimals. Since the difference between a variable and its limit is an infinitesimal, and since

$$\underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\Delta y}{\Delta x}\right]=f'(x);$$

$$\therefore \frac{\Delta y}{\Delta x}=f'(x)+\epsilon, \text{ or } \Delta y=f'(x)\Delta x+\epsilon\Delta x, \tag{1}$$

in which ϵ is an infinitesimal.

Now $$dy=f'(x)dx. \tag{2}$$

The value of dx being arbitrary, for convenience we shall, in this chapter, suppose it to be equal to Δx.

From (1) and (2) it follows that, if $dx=\Delta x=\iota$, dy and Δy are infinitesimals whose difference is infinitely small in comparison with either, and therefore, in differentiating, dy may be substituted for Δy (§ 205).

From these considerations we have the following simple rule for differentiating any function:

Find the increment of the function in terms of the increments of its variables, apply the principles of § 205, and in the terms remaining replace the increments by differentials.

Thus, to differentiate x^3, let $y = x^3$; then

$$\Delta y = 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3.$$

Hence, by the rule,

$$dy = 3x^2 dx.$$

Rem. We do not drop the infinitesimals of the higher orders, because they are nothing, or comparatively nothing, when added to an infinitesimal of the first order, but because we know that they do not appear in the limit of the ratio sought. Thus the method of limits is the *basis* of the method of infinitesimals, the difference being that in the latter we use infinitesimal differentials, and a quantity is dropped as soon as it appears, when it is known that it will vanish in passing to the limit sought. Any differential equation obtained by the infinitesimal method must evidently be true when the differentials are regarded as finite.

Examples.

1. Differentiate $u = xy$.

Here $\Delta u = y\Delta x + x\Delta y + \Delta x\Delta y$; $\therefore du = y\,dx + x\,dy$.

2. Differentiate $u = \frac{x}{y}$.

Here $\Delta u = \dfrac{y\Delta x - x\Delta y}{y^2 + y\Delta y}$; $\therefore du = \dfrac{y\,dx - x\,dy}{y^2}$.

3. Differentiate $y = \sin x$.

Here
$$\begin{aligned}\Delta y &= \sin(x + \Delta x) - \sin x\\ &= \cos x \sin \Delta x - (1 - \cos \Delta x)\sin x\\ &= \cos x\,\Delta x + v_2 i^2;\end{aligned}$$

for, when $\Delta x = \iota$, $\Delta x = \sin \Delta x + v_2\iota^2$ (§ 48, Cor., and § 205, Cor.), and $1 - \cos \Delta x = v_2\iota^2$ (§ 204, Ex. 3).

$$\therefore dy = \cos x\, dx.$$

4. Find the differential of any plane curve.

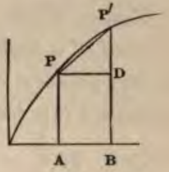

Fig. 58.

Let Δs, or arc PP′ in Fig. 58, be ι; then

$$\Delta s = \text{chord PP}' + v_2\iota^2 \qquad \S\ 48.$$
$$= \sqrt{\Delta x^2 + \Delta y^2} + v_2\iota^2;$$
$$\therefore ds = \sqrt{dy^2 + dx^2}.$$

5. Find the differential of the area between a curve and the axis of x.

Let Δx, or AB in Fig. 58, be ι; then, since area PDP′ $< \Delta x \cdot \Delta y$,

$$\Delta z = \text{area ABP}'\text{P} = y\,\Delta x + v_2\iota^2;$$
$$\therefore dz = y\, dx.$$

207. Second Fundamental Principle of Infinitesimals. Let $a_1, a_2, a_3, \cdots, a_n$ be any infinitesimals so related that, as n increases,

$$\text{limit}\,[a_1 + a_2 + a_3 + \cdots + a_n] = c;$$

and let $\beta_1, \beta_2, \beta_3, \cdots \beta_n$ be any other infinitesimals, such that

$$\frac{\beta_1}{a_1} = 1 + \epsilon_1,\ \frac{\beta_2}{a_2} = 1 + \epsilon_2,\ \cdots,\ \frac{\beta_n}{a_n} = 1 + \epsilon_n, \qquad (1)$$

in which $\epsilon_1, \epsilon_2, \cdots, \epsilon_n$ are infinitesimals.

Clearing equations (1) of fractions, adding, etc., we obtain

$$\beta_1 + \beta_2 + \cdots + \beta_n - (a_1 + a_2 + \cdots + a_n)$$
$$= a_1\epsilon_1 + a_2\epsilon_2 + \cdots + a_n\epsilon_n.$$

Let δ, a positive infinitesimal, be greater in absolute value than any of the infinitesimals $\epsilon_1, \epsilon_2, \cdots, \epsilon_n$; then we have numerically,

$$(\beta_1 + \beta_2 + \cdots + \beta_n) - (a_1 + a_2 + \cdots + a_n)$$
$$< \delta\,(a_1 + a_2 + \cdots + a_n).$$

But, since limit $\delta = 0$, and limit $(\alpha_1 + \alpha_2 + \cdots + \alpha_n) = c$,

$$\text{limit}\,[\delta(\alpha_1 + \alpha_2 + \cdots + \alpha_n)] = 0.$$

Whence $\text{limit}\,[\beta_1 + \beta_2 + \cdots + \beta_n] = \text{limit}\,[\alpha_1 + \alpha_2 + \cdots + \alpha_n].$

Hence, *if the difference between two infinitesimals is an infinitesimal of a higher order, either may be substituted for the other in any problem concerning the limit of the sum of infinitesimals, provided this limit is finite.*

COR. If $\dfrac{\beta_1}{\alpha_1} = 1 + \epsilon_1,\ \dfrac{\beta_2}{\alpha_2} = 1 + \epsilon_2,\ \cdots,\ \dfrac{\beta_n}{\alpha_n} = 1 + \epsilon_n,$

and $\alpha_1 + \alpha_2 + \cdots + \alpha_n = c,$

then $\text{limit}\,[\beta_1 + \beta_2 + \cdots + \beta_n] = \alpha_1 + \alpha_2 + \cdots + \alpha_n = c.$

208. Integration as a Summation. Let z represent the area between the curve OPD and the axis of x, and let us seek the area of the portion OBD. At the several values of x, as, 0, Oa, Oa', and Oa'', let

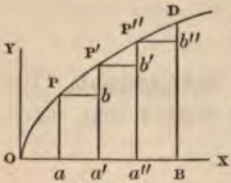

Fig. 59.

$$\text{O}a = aa' = a'a'' = a''\text{B} = \Delta x = dx;$$

then the corresponding values of dz are 0, $a\text{P}ba'$, $a'\text{P}'b'a''$, and $a''\text{P}''b''\text{B}$, while those of Δz are O Pa, aPP$'a'$, a'P$'$P$''a''$, and a''P$''$DB, whose sum equals OBD. Let the divisions Oa, aa', etc., become infinitesimals, but increase in number so that their sum will continually equal OB; then Δz and dz both become infinitesimals; and, since OPa, PP$'b$, etc., is each less than $\Delta x \cdot \Delta y$, $\Delta z - dz = v_2\iota^2$. Therefore the limit of the sum of the values of dz is equal to the sum of the values of Δz (§ 207, Cor.). Hence, if $x' = \text{OB}$, and the symbol $\sum\limits_{x=0}^{x=x'} \Delta z$ represent the sum of the values of Δz corresponding to the different values of x between 0 and x', we have

$$\text{area OBD} = \sum_{x=0}^{x=x'} \Delta z = \text{limit} \sum_{x=0}^{x=x'} dz = \text{limit} \sum_{x=0}^{x=x'} y\,dx.$$

But $$\text{area OBD} = \int_0^{x'} y\,dx;$$

$$\therefore \int_0^{x'} y\,dx = \text{limit} \sum_{x=0}^{x=x'} y\,dx.$$

Hence, *when differentials are infinitesimals, integration may be viewed as the summation of an infinite series of infinitesimals.*

209. Centre of Gravity. The *centre of gravity* of a body is a point so situated that, if it be supported, the body will remain at rest in whatever position it may be placed. An *element* of any quantity or magnitude is an infinitely small portion of it. The product of the weight of a body by the distance of its centre of gravity from a given plane is called the *moment* of the body with respect to that plane. The moment of a body is the sum of the moments of its elements. Hence the distance of the centre of gravity of a body from a given plane equals the sum of the moments of its elements divided by the weight of the body. The bodies here considered are supposed to be of uniform density; hence their weights are proportional to their volumes.

The advantage sometimes gained by viewing integration as a summation is illustrated in deducing formulas for finding the centre of gravity.

210. *To find the centre of gravity of any plane surface.* Let (x, y) be any point on the curve OP*n* referred to the axes OX and OY, and let x_0 and y_0 represent respectively the distances of the centre of gravity of any portion of the surface XO*n* from the planes OY and OX, which planes are perpendicular to that of the figure. The differential of the area XO*n* is $y\,dx$; now, if $dx = \text{AB} = \iota$, $y\,dx$ will differ from ABP′P, the corresponding element of this area, by $v_2\iota^2$; and the distance of the centre of gravity of this element from the plane OY will differ from x by $v_1\iota$; hence $xy\,dx$ will differ from the moment of this element with respect to the plane OY, by $v_2\iota^2$. Therefore, between $x = a$ and $x = b$, the sum of the moments of the elements

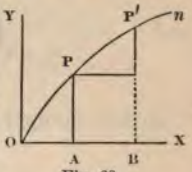

Fig. 60.

$$= \text{limit} \sum_{x=a}^{x=b} xy\,dx = \int_a^b xy\,dx;$$

$$\therefore x_0 = \frac{\int_a^b xy\,dx}{\text{area}} = \frac{\int_a^b xy\,dx}{\int_a^b y\,dx}.$$

Again, the centre of gravity of $y\,dx$ is evidently $\frac{1}{2}y$ from the plane OX; hence $\frac{1}{2}y^2dx$ will differ from the moment of the corresponding element with respect to the plane OX, by $v_2\iota^2$. Hence, between $x=a$ and $x=b$, the sum of the moments of the elements $=\frac{1}{2}\int_a^b y^2dx$;

$$\therefore y_0=\frac{1}{2}\frac{\int_a^b y^2dx}{\text{area}}=\frac{1}{2}\frac{\int_a^b y^2dx}{\int_a^b y\,dx}.$$

If the curve be symmetrical with respect to OX, x_0 is evidently the same for the whole area as for the half, and y_0 is zero.

211. *To find the centre of gravity of any plane curve.* Let (x_0, y_0) be the centre of gravity of any arc of a plane curve whose length is represented by s. Now, when $ds=\iota$, $x\,ds$ differs from the moment of the corresponding element of the curve, with respect to the plane OY (Fig. 60), by $v_2\iota^2$. Hence, between $x=a$ and $x=b$, the sum of the moments of the elements

$$=\int_a^b x\,ds,$$

and

$$x_0=\frac{\int_a^b x\,ds}{s}.$$

In like manner, we obtain

$$y_0=\frac{\int_a^b y\,ds}{s}.$$

212. *To find the centre of gravity of a solid of revolution.* The differential of a solid of revolution whose axis is the axis of x, is πy^2dx; hence, if $dx=\iota$, πxy^2dx will differ from the moment of the corresponding element of the solid, with respect to the plane OY, by $v_2\iota^2$; and therefore, between $x=a$ and $x=b$, the sum of the moments of the elements

$$=\int_a^b \pi xy^2dx.$$

$$\therefore x_0 = \frac{\pi \int_a^b xy^2 dx}{\text{volume}} = \frac{\int_a^b xy^2 dx}{\int_a^b y^2 dx}.$$

As the centre of gravity must evidently be on the axis of revolution, the formula given above entirely determines it.

EXAMPLES.

1. Determine the centre of gravity of a circular arc BAD.

Let the extremity D be (x', y').

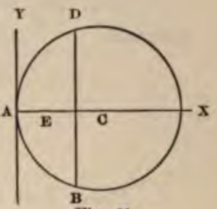

Fig. 61.

Here $\quad y^2 = 2rx - x^2$;

$$\therefore dy = \frac{(r-x)dx}{\sqrt{2rx - x^2}};$$

$$\therefore ds = \sqrt{dx^2 + dy^2} = \frac{rdx}{\sqrt{2rx - x^2}}.$$

$$\therefore \text{AE} = x_0 = \frac{\int_0^{x'} x\,ds}{s}$$

$$= \frac{r}{s}\int_0^{x'} \frac{x\,dx}{\sqrt{2rx - x^2}} = \frac{r}{s}\left[-\sqrt{2rx' - x'^2} + s\right]$$

$$= r - \frac{ry'}{s}.$$

Hence $\quad \text{CE} = r - \text{AE} = \dfrac{ry'}{s} = \dfrac{r \text{ chord BD}}{\text{arc BAD}}.$

2. Find the centre of gravity of a segment of a circle.

Using the equation of the circle referred to its centre, we have

$$x_0 = \frac{\int_a^b xy\,dx}{\text{area}} = \frac{\int_a^b (r^2 - x^2)^{\frac{1}{2}} x\,dx}{\text{area}}$$

$$= \frac{\frac{1}{3}(r^2 - a^2)^{\frac{3}{2}} - \frac{1}{3}(r^2 - b^2)^{\frac{3}{2}}}{\text{area}}.$$

If $a = 0$, and $b = r$, then area $= \frac{1}{4}\pi r^2$, and we have $x_0 = \dfrac{4r}{3\pi}$ when the segment is a semicircle.

3. **Find the centre of gravity of a parabolic area.**

Ans. $x_0 = \frac{3}{5}x'$.

4. **Find the centre of gravity of a right cone.**

Here $y = ax$, and volume $= \frac{1}{3}\pi y^2 x$;

$$\therefore x_0 = \frac{\pi\int_0^{x'} xy^2dx}{\text{volume}} = \frac{\pi\int_0^{x'} a^2x^3dx}{\frac{1}{3}\pi a^2x'^3} = \frac{3}{4}x';$$

that is, the distance of the centre of gravity from the vertex is three-fourths of the axis.

5. **Find the centre of gravity of a segment of a prolate spheroid.**

Ans. $x_0 = \frac{\frac{\pi b^2}{a^2}\left(\frac{2}{3}ax'^3 - \frac{1}{4}x'^4\right)}{\text{volume}}$.

When $x' = a$, $x_0 = \frac{5}{8}a$.

6. **Prove that** $x_0 = \frac{\int_a^b xyds}{\int_a^b yds}$ **is the formula for finding the centre of gravity of any surface of revolution.**